

Primal/Dual Method for the Grothendieck Constant

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Abstract

We determine the hundredths digit of the real Grothendieck constant by proving $1.773 \leq K_G \leq 1.7799$. Furthermore, numerical heuristics suggest the estimate $K_G \approx 1.779$. We start by simplifying the Gaussian duality theory of K_G , based on Hermite projection games and Krivine rounding schemes. The bounds are obtained by a heuristic primal/dual search for these objects, followed by rigorous *certification* of their values. The search and certification are performed using AI tools, and in particular the rigorous certificates are very large (analytical reductions to thousands of numerical inequalities).

1 Introduction

For a matrix $A \in \mathbb{R}^{m \times n}$, let

$$\text{OPT}(A) = \max_{x_i, y_j \in \{-1, 1\}} \sum_{i,j} A_{ij} x_i y_j, \quad \text{SDP}(A) = \sup_{\substack{u_i, v_j \in \mathbb{R}^{m+n} \\ \|u_i\| = \|v_j\| = 1}} \sum_{i,j} A_{ij} \langle u_i, v_j \rangle$$

The real Grothendieck constant K_G is defined by:

$$K_G = \sup_{m, n \in \mathbb{N}} \sup_{A \in \mathbb{R}^{m \times n}} \frac{\text{SDP}(A)}{\text{OPT}(A)}. \quad (1.1)$$

The quantity $\text{OPT}(A)$ is also called the “infinity-to-1 norm” of A . We refer to signs $x_i, y_j \in \{-1, 1\}$ achieving $\text{OPT}(A)$ as “infinity-to-1 optimizers” of A .

K_G is a basic mathematical constant arising in several fields including functional analysis [Pis12], quantum information [AGT06], and approximation algorithms [AN06, KN12]. The bound $K_G < \infty$ was first proven by Grothendieck and is now known as “Grothendieck’s inequality” [Gro56]. For many years it was conjectured that $K_G = \frac{\pi}{2 \ln(1+\sqrt{2})} \approx 1.7822$ until a breakthrough work by Braverman et al. [BMMN13] proved that this bound could be lowered by a positive $\varepsilon > 0$. Recently, these bounds moved by additional epsilons [Hei26a, JM26, Hei26b] until Saha et al. [SLX+26] proved the macroscopic bounds $1.7135 \leq \frac{6\pi}{11} \leq K_G \leq 1.7819$. Our main result is to prove that the first three digits of K_G are 1.77.

Theorem 1.1. $1.773 \leq K_G \leq 1.7799$

Our main conceptual innovation is a primal/dual method for algorithmically searching for upper and lower bounds on K_G . Running the method produces a pair of objects with nearly-matching value which heuristically suggest $K_G \approx 1.779$.

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To convert the primal/dual objects into rigorous bounds on K_G , we need to *certify* their values. Certification seems difficult: there is a gap between [Theorem 1.1](#) and the estimate $K_G \approx 1.779$ because we were not able to certify the instances found by the primal/dual search, instead using simpler instances for [Theorem 1.1](#). To explain the difficulties more, one of the two certification tasks is to prove an upper bound on $\text{OPT}(Q)$ for a candidate hard instance Q . While brute-force search can give us a good idea of the value of $\text{OPT}(Q)$, our proofs of $\text{OPT}(Q) \leq C$ are “brute-force proofs” involving extensive numerical inequalities proven using AI tools, which would have been impossible to obtain by hand.

The main body of the paper contains the general primal/dual theory, then outlines the instances and certificates used for [Theorem 1.1](#). The upper and lower bounds are proven in [Theorem 4.2](#) and [Theorem 4.1](#), respectively. The certificates are in the appendix.

1.1 Is there a short formula for K_G ?

Ideally, K_G would have a simple formula like $\sqrt{\pi}$ (this guess is now ruled out by [Theorem 1.1](#)). Below are formulas for three other constants related to quadratic optimization, which might suggest what a formula for K_G could look like.

Little Grothendieck constant. The little Grothendieck constant k_G is

$$k_G = \sup_{n \in \mathbb{N}} \sup_{A \in \mathbb{R}_{\text{sym}}^{n \times n}, A \succeq 0} \frac{\text{SDP}(A)}{\text{OPT}(A)} \quad (1.2)$$

where $\text{OPT}(A), \text{SDP}(A)$ are as in [Eq. \(1.1\)](#). It was proven by Grothendieck that $k_G = \frac{\pi}{2}$ [[Gro56](#)].

Goemans–Williamson constant. Let $\mathcal{L} = \bigcup_{n \in \mathbb{N}} \{L \in \mathbb{R}_{\text{sym}}^{n \times n} : L\mathbf{1} = 0, L \succeq 0, L_{ij} \leq 0 \text{ for } i \neq j\}$ be the set of Laplacian matrices for weighted graphs. The constant α_{GW} is¹

$$\alpha_{\text{GW}} = \inf_{L \in \mathcal{L}} \frac{\text{OPT}(L)}{\text{SDP}(L)}. \quad (1.3)$$

Then α_{GW} admits the well-known formula [[GW95](#), [FS02](#)]

$$\alpha_{\text{GW}} = \min_{t \in [-1, 1]} \frac{2 \arccos t}{\pi(1 - t)} \approx 0.87856.$$

The three constants $k_G, \alpha_{\text{GW}}, K_G$ are all integrality gaps of the SDP relaxation of $\text{OPT}(A)$, just restricted to different sets of matrices. The relatively simple formulas for k_G and α_{GW} can be traced to the simplicity of the optimal rounding algorithm (hyperplane rounding) for these special cases. Optimal lower bound instances are not too complicated either (Π_1 and the Feige-Schechtman instance [[FS02](#)], respectively). For K_G , these constructions are known not to achieve tight bounds, and the nearly-tight integrality gap instances we construct appear significantly more complicated.

Parisi constant. Another useful comparison is the Parisi constant P^* defined by [[Par79](#)]

$$P^* = \lim_{n \rightarrow \infty} n^{-3/2} \mathbb{E} \max_{x_i \in \{-1, 1\}} \sum_{i, j=1}^n A_{ij} x_i x_j \approx 1.5266 \quad (1.4)$$

¹When A is symmetric positive semidefinite, without loss of generality we can add the constraints $x_i = y_i$ to $\text{OPT}(A)$ and $u_i = v_i$ to $\text{SDP}(A)$. The standard definition of α_{GW} uses these symmetric variants.

where $A \in \mathbb{R}_{\text{sym}}^{n \times n}$ is random with $A_{ij} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1), i \leq j$ and $A_{ij} = A_{ji}$. Even though P^* is defined by a high-dimensional optimization problem, its value is given by the maximum value of a one-dimensional stochastic control problem [Mou25] (the famous *Parisi formula* is the dual problem, which expresses P^* as a minimization problem over nondecreasing functions $\gamma : [0, 1] \rightarrow [0, \infty)$).

The Parisi formula is only semi-explicit, since it involves an explicit convex optimization problem, but the optimizer γ apparently does not admit a simpler mathematical description besides being the solution to the optimization problem. Similarly, if there is very limited structure in optimal constructions for K_G , perhaps the best we can hope for is to express K_G as the value of a numerically tractable optimization problem.

2 Primal/dual method

(The proofs for this section are at the end of the section.)

2.1 Duality theory for K_G

It was known from prior work that two types of Gaussian objects are sufficient to precisely bound K_G from both sides, *Krivine rounding schemes* for the upper bound [NR14] and operators on Gaussian space for the lower bound [RS09] which we call *Hermite projection games* [JM26]. We start by giving a simpler, unified proof that these two classes of objects suffice using convex duality.

Let γ_n be n -dimensional standard Gaussian measure. A Hermite projection game is an operator on $L_2(\gamma_n)$ of the form, for some coefficients $q_m \in [-1, 1]$,

$$Q = \sum_{m=0}^{\infty} q_m \Pi_m^{(n)}$$

where $\Pi_m^{(n)}$ is the projection operator on $L_2(\gamma_n)$ to the span of the degree- m Hermite polynomials. We usually omit the superscript n in $\Pi_m^{(n)}$. Generalize to bounded operators on $L_2(\gamma_n)$ the notation,

$$\text{OPT}(Q) := \sup_{f, g: \mathbb{R}^n \rightarrow \{\pm 1\}} \langle f, Qg \rangle, \quad \text{SDP}(Q) := \sup_{k \in \mathbb{N}} \sup_{\substack{f, g: \mathbb{R}^n \rightarrow \mathbb{R}^k \\ \|f(X)\| = \|g(X)\| = 1}} \sum_{j=1}^k \langle f_j, Qg_j \rangle.$$

The operator Q is a witness for the lower bound $K_G \geq \text{SDP}(Q) / \text{OPT}(Q)$, after passing from operators to matrices by standard discretization arguments. We also write

$$v_n(q) = \text{OPT}(Q) = \sup_{f, g: \mathbb{R}^n \rightarrow \{\pm 1\}} \sum_{m=0}^{\infty} q_m \langle \Pi_m f, \Pi_m g \rangle, \quad v_{\infty}(q) = \sup_{n \in \mathbb{N}} v_n(q).$$

We will always take the limit $n \rightarrow \infty$ when calculating the lower bound proven by Q .

A Krivine rounding scheme is specified by a probability distribution μ on pairs of Boolean functions $f, g: \mathbb{R}^n \rightarrow \{-1, 1\}$. The rounding algorithm associated to μ is: given input unit vectors with arbitrary dimension, written $u, v \in \ell_2, \|u\| = \|v\| = 1$,

$$\text{sample } (f, g) \sim \mu \text{ and } G_{ij} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1) \text{ for } i \in [n], j \in \mathbb{N} \text{ then output } f(Gu), g(Gv) \in \{\pm 1\}. \quad (2.1)$$

The rounding algorithm for μ proves an upper bound on K_G in terms of an associated “kernel function” H_{μ} . For two inputs with $\langle u, v \rangle = t$ the expected correlation between the output signs is

$$H_{\mu}(t) := \mathbb{E}_{(f, g) \sim \mu} \mathbb{E}_{(X, Y) \sim t\text{-correlated Gaussians}} f(X)g(Y)$$

Let $c(\mu) = (c_m(\mu))_{m \in \mathbb{N}}$ be the Taylor coefficients of $H_\mu(t)$. The Taylor coefficients have the formula

$$H_\mu(t) = \sum_{m=0}^{\infty} c_m(\mu) t^m = \sum_{m=0}^{\infty} \mathbb{E}_{(f,g) \sim \mu} \langle \Pi_m f, \Pi_m g \rangle t^m.$$

We also write $c(f, g)$ and $H_{f,g}(t)$ for the case where μ is (f, g) with probability 1.

Lemma 2.1. *Let $n \in \mathbb{N}$ and let μ be a distribution on pairs of odd functions $f, g : \mathbb{R}^n \rightarrow \{\pm 1\}$. Then $K_G^{-1} \geq c_1(\mu) - \sum_{\text{odd } m \geq 3} |c_m(\mu)|$.*

We prove using convex duality that these two methods suffice to bound K_G from both sides.

Theorem 2.2. *Let $\mathcal{C} \subseteq \ell_1$ be the set of sequences $c(\mu)$, where $n \in \mathbb{N}$ and μ ranges over distributions on pairs of odd functions $f, g : \mathbb{R}^n \rightarrow \{\pm 1\}$. Let $\mathcal{Q} = \{q \in \ell_\infty : q_1 = 1, \forall m \in \mathbb{N}. |q_m| \leq 1, q_{2m} = 0\}$. Then $v_\infty(q) = \sup_{c \in \mathcal{C}} \langle c, q \rangle$ and*

$$\frac{1}{K_G} = \sup \left\{ c_1 - \sum_{\text{odd } m \geq 3} |c_m| : c \in \mathcal{C} \right\} = \inf \{ v_\infty(q) : q \in \mathcal{Q} \}. \quad (2.2)$$

Furthermore, the infimum over q is attained.

Corollary 2.3.

- (i) (Hermite projection games attain the lower bound [RS09])² There is $Q = \sum_{\text{odd } m \geq 1} q_m \Pi_m^{(n)}$ such that $q_1 = 1$ and $\text{SDP}(Q) / \text{OPT}(Q) = K_G - o_n(1)$.
- (ii) (Krivine schemes attain the upper bound [NR14]) For every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and a distribution μ on pairs of odd functions $f, g : \mathbb{R}^n \rightarrow \{-1, 1\}$ such that

$$c_1(\mu) - \sum_{\substack{m \geq 3 \\ m \text{ odd}}} |c_m(\mu)| \geq \frac{1}{K_G + \varepsilon}.$$

Via complementary slackness, convex duality implies that for any near-optimal pair of Q and μ , most of the probability mass of μ is on near-optimizers of Q . Hence hard lower bound instances Q are also the source of good rounding algorithms, via their infinity-to-1 optimizers.

Proposition 2.4. *Let $q \in \mathcal{Q}$ such that $v_\infty(q) = \frac{1}{K_G}$. Let μ be a distribution on $f, g : \mathbb{R}^n \rightarrow \{-1, 1\}$ such that $c_1(\mu) \geq \frac{1}{K_G} - \varepsilon$ and $\sum_{\text{odd } m \geq 3} |c_m(\mu)| \leq \varepsilon$. Then $\mathbb{E}_{(f,g) \sim \mu} \langle f, Qg \rangle \geq \frac{1}{K_G} - 2\varepsilon$.*

2.2 Numerical primal/dual search

By Theorem 2.2,

$$\frac{1}{K_G} = \sup_{c \in \mathcal{C}} \inf_{q \in \mathcal{Q}} \langle c, q \rangle = \inf_{q \in \mathcal{Q}} \sup_{c \in \mathcal{C}} \langle c, q \rangle \quad (2.3)$$

This suggests that we should search for extremizers c, q using a primal/dual algorithm, using the alternating update rules (assuming the extrema are attained),

$$q = \arg \min_{q \in \mathcal{Q}} \langle c, q \rangle, \quad c = \arg \max_{c \in \mathcal{C}} \langle c, q \rangle. \quad (2.4)$$

A very important caveat is that there is probably no efficient implementation of this algorithm. Calculating $\arg \max_{c \in \mathcal{C}} \langle c, q \rangle$ corresponds to calculating the infinity-to-1 optimizers of $Q = \sum_i q_i \Pi_i$ which is an NP-hard task in general. Therefore, we use heuristics to search for good infinity-to-1 optimizers of Q .

We use a slight variant of Eq. (2.4). Fix a parameter D . The iteration has two steps.

²Corollary 2.3 strengthens [RS09, Theorem 1.4] by proving that the infimum over $q \in \mathcal{Q}$ is attained.

1. In the first step, fix Q and heuristically compute its infinity-to-1 optimizers $(f_i, g_i)_{i \in I}$. Add $(f_i, g_i)_{i \in I}$ to a pool of rounding functions, denoted $\mathcal{R}$. Heuristically calculating the ∞ -to-1 optimizers is very slow and is guided by AI-heuristics.
2. In the second step, fix the pool $\mathcal{R}$ and re-compute Q to minimize the worst value among the rounding functions in the pool. Let $\mathcal{Q}_{\leq D} = \{q \in \mathcal{Q} : \forall m > D. q_m = 0\}$.

$$q = \arg \min_{q \in \mathcal{Q}_{\leq D}} \max_{(f, g) \in \mathcal{R}} \sum_{\substack{m=1 \\ m \text{ odd}}}^D q_m c_m(f, g) \quad Q = \sum_{m=1}^D q_m \Pi_m. \quad (2.5)$$

This computation can be formulated as a linear program parameterized by the coefficients $(c_m(f, g))_{1 \leq m \leq D}$ for $(f, g) \in \mathcal{R}$.

At the end of the iteration, calculate μ by the formula

$$\mu = \arg \max_{\text{supp}(\mu) \subseteq \mathcal{R}} c_1(\mu) - \sum_{m \geq 3} |c_m(\mu)|. \quad (2.6)$$

This can be formulated as a linear program in μ again parameterized by $(c_m(f, g))_{1 \leq m \leq D}$.

Running the algorithm leads to a pair of candidates with heuristic primal and dual estimates of K_G matching on the first four digits $K_G \approx 1.779$. See [Section A](#) for numerical details of the run.

2.3 Proofs for the duality theory

Lemma 2.5. *Let $f, g : \mathbb{R}^n \rightarrow \{\pm 1\}$ be measurable. Then $\sum_m |c_m(f, g)| \leq 1$. Thus, $\mathcal{C} \subseteq \ell_1$. The series for $H_{f, g}$ and H_μ converge absolutely and uniformly for $t \in \mathbb{C}, |t| \leq 1$ and are holomorphic for $|t| < 1$.*

Proof. Recall $c_i(f, g) = \langle \Pi_i f, \Pi_i g \rangle$. Using Cauchy–Schwarz twice,

$$\sum_k |\langle \Pi_k f, \Pi_k g \rangle| \leq \sum_k \|\Pi_k f\|_2 \|\Pi_k g\|_2 \leq \left(\sum_k \|\Pi_k f\|_2^2 \right)^{1/2} \left(\sum_k \|\Pi_k g\|_2^2 \right)^{1/2} = \|f\|_2 \|g\|_2 = 1.$$

Then for a general μ we have $\sum_m |c_m(\mu)| \leq \mathbb{E}_{(f, g) \sim \mu} \sum_m |c_m(f, g)| \leq 1$. Absolute summability gives the stated convergence and holomorphy. $\square$

Lemma 2.6. *Let $q \in \mathcal{Q}$ and $Q = \sum_{\text{odd } m \geq 1} q_m \Pi_m$. Then*

$$\text{OPT}(Q) = \sup_{f, g : \mathbb{R}^n \rightarrow \{\pm 1\}} \langle f, Qg \rangle = \sup_{\text{odd } f, g : \mathbb{R}^n \rightarrow \{\pm 1\}} \langle f, Qg \rangle.$$

Proof. Given $f, g : \mathbb{R}^n \rightarrow \{\pm 1\}$ their odd parts $f_{\text{odd}}(X) = \frac{f(X) - f(-X)}{2}$ and $g_{\text{odd}}(X) = \frac{g(X) - g(-X)}{2}$ satisfy $\langle f, Qg \rangle = \langle f_{\text{odd}}, Qg_{\text{odd}} \rangle$. Furthermore, $f_{\text{odd}}(X), g_{\text{odd}}(X) \in \{-1, 0, 1\}$.

Fix any odd Boolean function a , for example $a(X) = \text{sign}(X_1)$. Define

$$f_{\pm} = f_{\text{odd}} \pm a \mathbf{1}_{\{f_{\text{odd}}=0\}}, \quad g_{\pm} = g_{\text{odd}} \pm a \mathbf{1}_{\{g_{\text{odd}}=0\}}.$$

These functions are odd and Boolean, and their respective averages are f_{odd} and g_{odd} . Hence, $\langle f_{\pm}, Qg_{\pm} \rangle \geq \langle f_{\text{odd}}, Qg_{\text{odd}} \rangle$ for at least one of the four choices. Taking suprema proves the claim. $\square$

Proof of Lemma 2.1. Let $A \in \mathbb{R}^{m \times n}$ and let $u_i, v_j \in \ell_2, \|u_i\| = \|v_j\| = 1$ be an optimal SDP solution. Taking tensor powers shows

$$\left| \sum_{i,j} A_{ij} \langle u_i, v_j \rangle^m \right| = \left| \sum_{i,j} A_{ij} \langle u_i^{\otimes m}, v_j^{\otimes m} \rangle \right| \leq \text{SDP}(A) \quad (m \geq 1). \quad (2.7)$$

Let $\tilde{x}_i, \tilde{y}_j \in \{\pm 1\}$ be the randomized signs output by Gaussian rounding in Eq. (2.1). We have

$$\text{OPT}(A) \geq \sum_{i,j} A_{ij} \mathbb{E} \tilde{x}_i \tilde{y}_j = \sum_{i,j} A_{ij} H_\mu(\langle u_i, v_j \rangle) \stackrel{\text{Eq. (2.7)}}{\geq} \left(c_1(\mu) - \sum_{\substack{m \geq 3 \\ m \text{ odd}}} |c_m(\mu)| \right) \text{SDP}(A).$$

Therefore $\frac{\text{OPT}(A)}{\text{SDP}(A)} \geq c_1(\mu) - \sum_{\text{odd } m \geq 3} |c_m(\mu)|$ for all $A \neq 0$ which implies the same bound on K_G^{-1} . $\square$

Proof of Theorem 2.2. By Lemma 2.6, $v_\infty(q) = \sup_{c \in \mathcal{C}} \langle c, q \rangle$. We prove

$$\frac{1}{K_G} \leq \inf \{v_\infty(q) : q \in \mathcal{Q}\} = \sup \left\{ c_1 - \sum_{\text{odd } m \geq 3} |c_m| : c \in \mathcal{C} \right\} \leq \frac{1}{K_G}.$$

The right inequality is Lemma 2.1. The middle equality is convex duality. Indeed, $c_1 - \sum_{\text{odd } m \geq 3} |c_m| = \inf_{q \in \mathcal{Q}} \langle c, q \rangle$ for all $c \in \ell_1$. This implies the weak duality $\sup_{c \in \mathcal{C}} \inf_{q \in \mathcal{Q}} \langle c, q \rangle \leq \inf_{q \in \mathcal{Q}} \sup_{c \in \mathcal{C}} \langle c, q \rangle$. Strong duality holds by Sion's minimax theorem: $\mathcal{C}$ is convex, $\mathcal{Q}$ is convex and compact in the weak-* topology on $\ell_\infty = (\ell_1)^*$, and $\langle c, q \rangle$ is affine and continuous in each variable separately.

For the left inequality, let $q \in \mathcal{Q}$ and let $Q = \sum_m q_m \Pi_m^{(n)}$. Then $\text{OPT}(Q) = v_n(q) \leq v_\infty(q)$. On the other hand, $\text{SDP}(Q) \geq 1 - o_n(1)$ by $q_1 = 1$ and a standard estimate of the SDP value of the vector solution $f(x) = g(x) = \frac{x}{\|x\|}$. Thus $\text{SDP}(Q)/\text{OPT}(Q) \geq \frac{1 - o_n(1)}{v_\infty(q)}$ and $K_G \geq \frac{1}{v_\infty(q)}$ as desired.

We have that $v_\infty(q) = \sup_{c \in \mathcal{C}} \langle c, q \rangle$ is weak-* lower semicontinuous since it is a supremum of weak-* continuous functions. Then it attains a minimum on the weak-* compact set $\mathcal{Q}$. $\square$

Proof of Corollary 2.3. (Hermite projection games) Let $q \in \mathcal{Q}$ attain the infimum in Eq. (2.2). Then $Q = \sum_m q_m \Pi_m^{(n)}$ has $\text{SDP}(Q) \geq 1 - o_n(1)$ as in the previous proof, and $\text{OPT}(Q) = v_n(q) \leq v_\infty(q)$. Therefore $\text{SDP}(Q)/\text{OPT}(Q) \geq \frac{1 - o_n(1)}{v_\infty(q)} = K_G - o_n(1)$.

(Krivine rounding schemes) By definition of $\mathcal{C}$ and Theorem 2.2. $\square$

Proof of Proposition 2.4. The assumptions on μ with $q_1 = 1, |q_m| \leq 1$ imply $\langle c(\mu), q \rangle = \sum_i c_i(\mu) q_i \geq K_G^{-1} - 2\varepsilon$. On the other hand,

$$\sum_i c_i(\mu) q_i = \mathbb{E}_{(f,g) \sim \mu} \sum_i q_i \langle \Pi_i f, \Pi_i g \rangle = \mathbb{E}_{(f,g) \sim \mu} \langle f, Qg \rangle.$$

We conclude that $\mathbb{E}_{(f,g) \sim \mu} \langle f, Qg \rangle \geq K_G^{-1} - 2\varepsilon$. $\square$

3 Asymptotic Boolean functions

In this section, we lift Hermite projection games and Krivine rounding schemes to an explicit asymptotic space in the limit $n \rightarrow \infty$. We call “asymptotic Boolean functions” the limits of functions $f : \mathbb{R}^n \rightarrow \{-1, 1\}$. These frequently arise as ∞ -to-1 optimizers and therefore are also used in the numerical primal/dual search. Asymptotic Boolean functions are closely related to the “limiting Krivine schemes” used by Saha et al. [SLX⁺26].

Example 3.1. Consider the Hermite projection matrix $\Pi_d^{(n)}$ with $d > 1$. As $n \rightarrow \infty$, the ∞ -to-1 norm of Π_d is asymptotically achieved only by functions $f, g : \mathbb{R}^n \rightarrow \{\pm 1\}$ using a diverging number of variables, such as

$$f(X) = g(X) = \text{sign} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \text{He}_d(X_i) \right) \quad (3.1)$$

where He_d are the Hermite polynomials. It can be shown that f, g in Eq. (3.1) are asymptotic optimizers of $\text{OPT}(Q)$. Calculating their value shows $\text{SDP}(\Pi_d) / \text{OPT}(\Pi_d) = \frac{\pi}{2} - o_n(1)$. By the little Grothendieck inequality, $\frac{\pi}{2}$ is the maximum possible integrality gap for PSD matrices. The standard construction for little Grothendieck uses $d = 1$ but it turns out that arbitrary d has the same gap.

We use the following standard notions from Gaussian analysis. Let X be a set of variables. Sample $X_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Let He_k be the unnormalized probabilists' Hermite polynomials. Let $\psi_k = \text{He}_k / \sqrt{k!}$ be their normalizations. For a finitely-supported multi-index $\alpha \in \mathbb{N}^X$ let $X^\alpha = \prod_i X_i^{\alpha_i}$ and similarly for He_α and ψ_α . Let $|\alpha| = \sum_i \alpha_i$ and $\alpha! = \prod_i \alpha_i!$. Then $\{\psi_\alpha(X) : \alpha \in \mathbb{N}^X, |\alpha| < \infty\}$ is an orthonormal basis of $L_2(X)$. Let $\hat{f}_\alpha = \mathbb{E}[f\psi_\alpha]$. Let $\mathcal{H}_d(X)$ be the closed span of the degree- d Hermite polynomials in X , which is also called the “degree- d Gaussian chaos” for X .

3.1 Graded Gaussian space

The idea of the asymptotic limit is to replace the summation in Eq. (3.1) by a new Gaussian variable, since it is asymptotically Gaussian. In the case of Eq. (3.1), we assign the new Gaussian variable “grade d ” to represent that it is a degree- d polynomial in the “original” variables.

Let X be a set of variables such that each variable has an associated positive integer *grade*. Sample $X_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$ independently across all variables regardless of grade. We will notate variables in X as $X_i^{(d)}$ where d indicates the grade.

Let $\text{gr}(\alpha)$ be the total grade of a multi-index $\alpha \in \mathbb{N}^X$. Polynomials in X now have two notions of degree, their standard degree, and their graded-degree, or total grade. The Hermite polynomial $\text{He}_\alpha(X)$ is defined as before, and $\psi_\alpha(X)$ are still an orthonormal basis for $L_2(X)$.

The following notions behave differently on the graded space. Let Π_i be the operator on $L_2(X)$ which projects to the span of graded-degree i Hermite polynomials. On the graded space, oddness means $f(\tilde{X}) = -f(X)$ under the graded sign transformation $\tilde{X}_i^{(d)} = (-1)^d X_i^{(d)}$. An odd function satisfies $\Pi_{2m}f = 0$ since by oddness and $\psi_\alpha(\tilde{X}) = (-1)^{\text{gr}(\alpha)}\psi_\alpha(X)$ we have for all even $\text{gr}(\alpha)$,

$$\mathbb{E}[f(X)\psi_\alpha(X)] = -(-1)^{\text{gr}(\alpha)} \mathbb{E}[f(X)\psi_\alpha(X)] = 0. \quad (3.2)$$

Let $W_d(f) = \|\Pi_d f\|_2^2$. For variables with grade k , the Ornstein–Uhlenbeck noise operator acts by sampling t -correlated Gaussians Y with covariance $\mathbb{E}[X_i Y_i] = t^k$. Under these definitions, we still have the noise correlation and Parseval identities,

$$\mathbb{E}_{(X,Y) \sim t\text{-correlated}}[f(X)g(Y)] = \sum_{d \geq 0} t^d \langle \Pi_d f, \Pi_d g \rangle, \quad \mathbb{E}[f(X)^2] = \sum_{d \geq 0} W_d(f). \quad (3.3)$$

Now set X to be a collection of countably many variables of each graded degree. Write $X_1, X_2, X_3, \dots$ for the grade-1 variables in X and $X_1^{(d)}, X_2^{(d)}, X_3^{(d)}, \dots$ for the (independent) grade- d variables in X . Define the asymptotic limit spaces

$$\overline{\mathcal{H}}_d = \overline{\text{span}}(\text{He}_\alpha(X) : \text{gr}(\alpha) = d), \quad \overline{L}_2 = L_2(X). \quad (3.4)$$

The notation $\overline{\text{span}}$ in Eq. (3.4) denotes the closure of the linear span in $L_2(X)$ which is distinct from the notation $\overline{\mathcal{H}}_d$ and $\overline{L}_2$. We write either f or $f(X)$ for $f \in \overline{L}_2$.

As shown in [Example 3.1](#), these spaces contain natural limits of Gaussian functions. More interestingly, in the sense of laws, $\overline{\mathcal{H}}_d$ contains *all possible* limits of degree- d Gaussian polynomials. We prove this in another paper by one of the current authors [\[Jon26\]](#).

Theorem 3.2 ([\[HMP24, Jon26\]](#)). *Let $d_i \in \mathbb{N}$ for $i \in \mathbb{N}$.*

- (i) *(Closure of the asymptotic space) Let $p_{n,i} \in \overline{\mathcal{H}}_{d_i}$ such that $\sup_n \|p_{n,i}\|_2 < \infty$ for all i . Then there exists a subsequence and $p_i \in \overline{\mathcal{H}}_{d_i}$ such that, along the subsequence, $(p_{n,i})_{i \in \mathbb{N}} \xrightarrow{d} (p_i)_{i \in \mathbb{N}}$ with convergence of all mixed (finite) moments. Consequently, $\{\text{Law}(p) : p \in \overline{\mathcal{H}}_d\}$ is weakly closed.*
- (ii) *(Approximability by finitary functions) For all $p_i \in \overline{\mathcal{H}}_{d_i}$ there exists a sequence $p_{n,i} \in \mathcal{H}_{d_i}(X_1, \dots, X_n)$ such that $(p_{n,i})_{i \in \mathbb{N}} \xrightarrow{d} (p_i)_{i \in \mathbb{N}}$ with convergence of all mixed (finite) moments.*

As a corollary, $\overline{\mathcal{L}}_2$ contains all limits of Gaussian functions with uniform tail bounds on their Hermite coefficients.

3.2 Asymptotic Boolean functions on Gaussian space

An asymptotic Boolean function is $f \in \overline{\mathcal{L}}_2$ such that $f \in \{-1, 1\}$ almost surely. While this contains all finitary Boolean functions $f : \mathbb{R}^n \rightarrow \{-1, 1\}$, it also contains asymptotic functions which use graded variables, such as $f(X) = \text{sign}(X_1^{(2)})$ and $f(X) = \text{sign}(X_1^{(1)}X_1^{(2)} - X_1^{(3)})$.

Extend the notations $c(\mu), c(f, g), H_\mu(t), H_{f,g}(t)$ using $c_m(f, g) = \langle \Pi_m f, \Pi_m g \rangle$. The proof of [Lemma 2.5](#) still gives $\sum_m |c_m(\mu)| \leq 1$, and hence uniform convergence of H_μ on $|t| \leq 1$ and holomorphy on $|t| < 1$.

In the remainder of the section, we prove several [Lemmas 3.3, 3.4, 3.5](#) showing that maximizers are attained within the class of asymptotic Boolean functions.

Lemma 3.3. *Let $\mathcal{K} \subseteq \ell_1$ be the set of $c(\mu)$ for distributions μ on pairs of odd $f, g \in \overline{\mathcal{L}}_2, f(X), g(X) \in \{\pm 1\}$. Then $\mathcal{K} = \overline{\mathcal{C}}$ where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in ℓ_1 .*

Proof. We use the auxiliary set

$$\mathcal{B} = \{c(f, g) : f, g \in \overline{\mathcal{L}}_2 \text{ odd}, |f|, |g| \leq 1\}.$$

First we claim $\mathcal{B} \subseteq \mathcal{K}$. Let $f, g \in \overline{\mathcal{L}}_2$ odd, $|f|, |g| \leq 1$. We use an odd version of threshold rounding. Let a be any odd Boolean function, such as $a(X) = \text{sign } X_1$. For $s \in [-1, 1]$, define

$$R_s h(X) = a(X) \text{sign}(a(X)h(X) - s),$$

with $\text{sign}(0) = 1$. If h is odd and bounded by 1, then $R_s h$ is odd and Boolean, and $\mathbb{E}_s R_s h = h$ for uniform $s \in [-1, 1]$. Thus, for independent uniform s, t we have

$$\mathbb{E}_{s,t} c(R_s f, R_t g) = c(f, g). \tag{3.5}$$

Thus $\mathcal{B} \subseteq \mathcal{K}$. The same proof shows that if odd $|f|, |g| \leq 1$ depend on only finitely many grade-one variables, then $c(f, g) \in \mathcal{C}$.

We next prove compactness of $\mathcal{B}$. Take a sequence of odd $(f_n, g_n), |f_n|, |g_n| \leq 1$. By [Theorem 3.2\(i\)](#), after passing to a subsequence there are $p_d, r_d \in \overline{\mathcal{H}}_d$ such that

$$(\Pi_d f_n, \Pi_d g_n)_{d \in \mathbb{N}} \xrightarrow{d} (p_d, r_d)_{d \in \mathbb{N}} \tag{3.6}$$

with convergence of the mixed moments of every finite subtuple. For each fixed D ,

$$\sum_{d \leq D} \|p_d\|_2^2 = \lim_{n \rightarrow \infty} \sum_{d \leq D} \|\Pi_d f_n\|_2^2 \stackrel{\text{Parseval}}{\leq} 1.$$

Taking sup over D shows $\sum_d \|p_d\|_2^2 \leq 1$. So $f = \sum_d p_d$ is well-defined in $\overline{L_2}$ and $\Pi_d f = p_d$. The same holds for g . Both are odd since their even-degree components vanish.

Note that we may *not* have $f_n \xrightarrow{d} f$ since Hermite mass for f_n could escape to higher and higher degree. We still prove $|f|, |g| \leq 1$. Fix $0 < r < 1$ and define the graded Ornstein–Uhlenbeck operator $T_r = \sum_d r^d \Pi_d$. Eq. (3.6) together with

$$\sup_n \left\| T_r f_n - \sum_{d \leq D} r^d \Pi_d f_n \right\|_2 \leq r^{D+1},$$

implies $T_r f_n \xrightarrow{d} T_r f$. For the nonnegative continuous function $\varphi(x) = (|x| - 1)_+$, the Portmanteau inequality gives

$$\mathbb{E} \varphi(T_r f) \leq \liminf_{n \rightarrow \infty} \mathbb{E} \varphi(T_r f_n) = 0,$$

since $|T_r f_n| \leq 1$. Finally, $T_r f \rightarrow f$ in L_2 as $r \uparrow 1$, and φ is 1-Lipschitz, so $\mathbb{E} \varphi(f) = 0$. Hence $|f| \leq 1$ almost surely. The argument for g is identical.

We have

$$c_d(f_n, g_n) = \mathbb{E}[(\Pi_d f_n)(\Pi_d g_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[(\Pi_d f)(\Pi_d g)] = c_d(f, g)$$

Thus every sequence in $\mathcal{B}$ has a coordinatewise convergent subsequence. Since the coordinatewise topology is metrizable, sequential compactness implies $\mathcal{B}$ is (topologically) compact.

We now prove $\mathcal{B} \subseteq \overline{\mathcal{C}}$. Fix odd (f, g) , $|f|, |g| \leq 1$, and put

$$f^{(D)} = \sum_{d \leq D} \Pi_d f, \quad g^{(D)} = \sum_{d \leq D} \Pi_d g, \quad \varepsilon_D = \|f - f^{(D)}\|_2 + \|g - g^{(D)}\|_2.$$

Then $\varepsilon_D \rightarrow 0$ as $D \rightarrow \infty$. Theorem 3.2 (ii) gives odd finite-dimensional polynomials F_n, G_n of degree at most D such that, also with convergence of all mixed moments,

$$(\Pi_d F_n, \Pi_d G_n)_{d \leq D} \xrightarrow{d} (\Pi_d f, \Pi_d g)_{d \leq D}.$$

In particular,

$$c(F_n, G_n) \rightarrow c(f^{(D)}, g^{(D)}) \quad \text{in } \ell_1, \quad \|F_n\|_2 \rightarrow \|f^{(D)}\|_2 \leq 1,$$

and similarly for G_n .

Let $\chi(u) = \max\{-1, \min\{u, 1\}\}$ be the clipping function to $[-1, 1]$. Convergence in distribution and uniformly bounded fourth moments imply

$$\begin{aligned} \|F_n - \chi(F_n)\|_2 &\rightarrow \|f^{(D)} - \chi(f^{(D)})\|_2 \leq \|f^{(D)} - f\|_2, \\ \|G_n - \chi(G_n)\|_2 &\rightarrow \|g^{(D)} - \chi(g^{(D)})\|_2 \leq \|g^{(D)} - g\|_2. \end{aligned} \tag{3.7}$$

The right inequalities use that clipping is the pointwise projection onto $[-1, 1]$. Orthogonality and Cauchy–Schwarz give the general continuity relations,

$$\begin{aligned} \|c(f, g)\|_1 &\leq \|f\|_2 \|g\|_2, \\ \|c(f, g) - c(f', g')\|_1 &\leq \|f - f'\|_2 \|g\|_2 + \|f'\|_2 \|g - g'\|_2. \end{aligned} \tag{3.8}$$

Using Eq. (3.7) and (3.8), we obtain

$$\limsup_{n \rightarrow \infty} \|c(\chi(F_n), \chi(G_n)) - c(f, g)\|_1 \leq 2\varepsilon_D.$$

The clipped functions are bounded by 1, odd, and finite-dimensional, so their coefficient sequences belong to $\mathcal{C}$ by the sentence after (3.5). Letting $D \rightarrow \infty$ proves the inclusion.

It remains to pass from $\mathcal{B}$ to $\mathcal{K}$. The set $\mathcal{K}$ is exactly the set of barycenters of probability measures on $\mathcal{B}$. Indeed, if $c(\mu) \in \mathcal{K}$, then $c(\mu) = \mathbb{E}_{(f,g) \sim \mu} c(f,g)$ is also a mixture of elements $c(f,g) \in \mathcal{B}$. Conversely, a probability measure on $\mathcal{B}$ can be lifted³ to a measure μ on odd pairs $(f,g), |f|, |g| \leq 1$, and then threshold rounding in (3.5) ensures $f, g \in \{\pm 1\}$ without changing $c(\mu)$.

Probability measures on the compact metric space $\mathcal{B}$ form a compact space for weak convergence. Their barycenters depend continuously on each coordinate, so $\mathcal{K}$ is coordinatewise compact and convex.

The inclusion map $\mathcal{B} \rightarrow \ell_1$ is Borel measurable, and $\|c\|_1 \leq 1$ on $\mathcal{B}$, so these barycenters are also Bochner integrals in ℓ_1 . Since $\overline{\mathcal{C}}$ is closed and convex and contains $\mathcal{B}$, it contains every such barycenter. Hence $\mathcal{K} \subseteq \overline{\mathcal{C}}$. Conversely, $\mathcal{C} \subseteq \mathcal{K}$, and coordinatewise compactness makes $\mathcal{K}$ closed in ℓ_1 . Therefore $\mathcal{K} = \overline{\mathcal{C}}$. $\square$

We prove that the asymptotic optimum of a finitely-supported Hermite projection game is achieved by an asymptotic Boolean function. We call these “asymptotic ∞ -to-1 optimizers” of Q . Moreover, there exist optimizers which are $\overline{\mathcal{H}}_{\leq d}$ threshold functions (essentially, polynomial threshold functions).

Lemma 3.4. *Let $q \in \ell_\infty$ with $q_{2m} = 0$ for $m \geq 0$ and assume $\max \text{supp}(q) = D < \infty$. Then*

$$v_\infty(q) = \max \left\{ \sum_{m=0}^{\infty} q_m \langle \Pi_m f, \Pi_m g \rangle : f, g \in \overline{L_2}, f(X), g(X) \in \{\pm 1\} \text{ a.s.} \right\}.$$

Furthermore, there exist optimizers of the form $f = \text{sign}(p)$ and $g = \text{sign}(r)$ for $p, r \in \overline{\mathcal{H}}_{\leq D}$.

Proof. Let $Q = \sum_d q_d \Pi_d$ on $\overline{L_2}$. First we claim (this still holds if $\max \text{supp}(q) = \infty$)

$$\text{OPT}(Q) = \sup_{c \in \mathcal{B}} \langle c, q \rangle = v_\infty(q). \quad (3.9)$$

The first equality follows since taking the odd parts of candidates f, g for $\text{OPT}(Q)$ preserves their value, and $f, g \in [-1, 1]$ can be replaced by a distribution of $f, g \in \{\pm 1\}$ using the threshold rounding in (3.5). For the second equality, the inclusion in Lemma 3.3 and ℓ_1 -continuity of the pairing give the upper bound $\sup_{c \in \mathcal{B}} \langle c, q \rangle \leq \sup_{c \in \mathcal{C}} \langle c, q \rangle = v_\infty(q)$. For the reverse inequality, a sequence of $c(\mu^{(k)}) \in \mathcal{C}$ approaching $v_\infty(q)$ has that $c(\mu^{(k)}) = \mathbb{E}_{(f,g) \sim \mu^{(k)}} c(f,g)$ is an average of $c(f,g) \in \mathcal{B}$, so the inequality follows.

When q has finite support, the map $c \mapsto \langle c, q \rangle$ is coordinatewise continuous. Compactness of $\mathcal{B}$ and (3.9) therefore give $c \in \mathcal{B}$ achieving value $v_\infty(q)$. Then any pair $(f,g), |f|, |g| \leq 1$ corresponding to c achieves the maximum value $v_\infty(q)$.

To see that f, g can be polynomial threshold functions, take two successive best responses:

$$p = Qg, \quad f = \text{sign}(p), \quad r = Qf, \quad g = \text{sign}(r).$$

Neither replacement decreases the value, and (3.9) says that no Boolean pair can exceed $v_\infty(q)$. Hence (f,g) is optimal. Both p and r have graded degree at most D . $\square$

Lemma 3.5.

- (i) *There is an operator $Q = \sum_{\text{odd } m \geq 1} q_m \Pi_m$ on $\overline{L_2}$ such that $q_1 = 1$ and $\text{SDP}(Q)/\text{OPT}(Q) = K_G$.*
- (ii) *There is a distribution μ supported on odd functions $f, g \in \overline{L_2}, f(X), g(X) \in \{-1, 1\}$ such that $H_\mu(t) = \frac{t}{K_G}$.*

³The space of bounded odd pairs is a closed subset of $\overline{L_2}^2$, hence Polish. Its coefficient map onto $\mathcal{B}$ is continuous by (3.8). The Jankov-von Neumann theorem gives a section measurable for the completion of the given probability measure, which supplies the lift.

Proof. (i) Choose $q \in \mathcal{Q}$ attaining the infimum in [Theorem 2.2](#), so $v_\infty(q) = K_G^{-1}$. For $Q = \sum_d q_d \Pi_d$, (3.9) gives $\text{OPT}(Q) = K_G^{-1}$. Its operator norm is at most 1, so $\text{SDP}(Q) \leq 1$. The standard SDP construction $f(X) = g(X) = (X_1, \dots, X_n) / \sqrt{X_1^2 + \dots + X_n^2}$ with $n \rightarrow \infty$ shows $\text{SDP}(Q) \geq q_1 = 1$. Therefore $\text{SDP}(Q) / \text{OPT}(Q) = K_G$.

(ii) For every finitely supported real sequence b supported on odd degrees, Grothendieck's inequality gives

$$\sup_{c \in \mathcal{B}} \langle c, b \rangle = v_\infty(b) \geq \text{OPT}(Q_b) \geq K_G^{-1} \text{SDP}(Q_b)$$

where $Q_b = \sum_m b_m \Pi_m$. By the standard SDP construction, $\text{SDP}(Q_b) \geq b_1$. Therefore, for all such b ,

$$\sup_{c \in \mathcal{B}} \langle c, b \rangle \geq K_G^{-1} b_1.$$

Because $\mathcal{K}$ is coordinatewise compact and convex, finite-dimensional separation implies $K_G^{-1} e_1 \in \mathcal{K}$. By the definition of $\mathcal{K}$, there is a distribution μ on odd asymptotic Boolean pairs with $c(\mu) = K_G^{-1} e_1$. Hence $H_\mu(t) = t/K_G$ on $[-1, 1]$. $\square$

4 Overview of the rigorous certificates

4.1 Certifying the lower bound

To prove a lower bound on K_G , we prove an upper bound on $\text{OPT}(Q)$ for a Hermite projection game $Q = \sum_m q_m \Pi_m$.⁴ Set $q_m = 0$ except at the following degrees; every decimal is exact:

m	q_m	m	q_m
1	1	13	0.072282834573
3	-0.654509751718	15	-0.004420811526
5	-0.159925966134	17	-0.058106583958
7	0.047785078787	19	-0.041412929910
9	0.086465878916	21	0.016668224599
11	0.076371662489		

(4.1)

Theorem 4.1. $v_\infty(q) \leq \frac{1000}{1773}$. Therefore, $K_G \geq 1.773$.

The proof of [Theorem 4.1](#) is in [Section B](#). Proposition [B.1](#) reduces estimating $\text{OPT}(Q)$ to a 44-dimensional optimization problem, which is then reduced to checking a finite number of inequalities numerically. The code for the numerical portions is available at <https://github.com/christophsjones/RealGrothendieck>.

4.2 Certifying the upper bound

To prove an upper bound on K_G , we estimate $c_1(\mu) - \sum_{\text{odd } m \geq 3} |c_m(\mu)|$ for a mixture of rounding functions μ .

Parameters. The rounding scheme is parametrized by 31 polynomials, each with a weight associated to it, whose coefficients and weights are specified in [Table 1](#). The additional weight for $i = 14$ corresponds to a function specified below. For $i \in \{0, \dots, 31\}$, let λ_i be the *normalized* weight. For

⁴The lower bound of [\[SLX+26\]](#) can be interpreted as certifying the tight inequality $\text{OPT}(\Pi_1 - 0.5\Pi_3) \leq \frac{11}{6\pi}$.

$i \neq 14$, let P_i be the i -th polynomial normalized by the sum of the absolute values of its coefficients (if this sum is smaller than 1, P_i is left unnormalized), so that

$$\sum_{i=0}^{31} \lambda_i = 1, \quad \lambda_i > 0, \quad \sum_{m \geq 1} |[t^m]P_i(t)| \leq 1 \quad (i \neq 14) \quad (4.2)$$

where $[t^m]$ denotes the coefficient of the monomial t^m for the i -th polynomial. Define

$$F(w, x_1, x_2, x_3) := \text{sign} \left(w + \sum_{j=1}^3 r(x_j) \right), \quad (4.3)$$

where

$$\begin{aligned} r(s) := & -0.002608895910895478\psi_1(s) - 0.0999107408341197\psi_3(s) \\ & - 0.020910560460516325\psi_5(s) - 0.005853637316670958\psi_7(s) \\ & + 0.0057627068821593744\psi_9(s) + 0.0013019893459673402\psi_{11}(s) \\ & - 0.0019418422133255495\psi_{13}(s) + 0.0006843699264111792\psi_{15}(s). \end{aligned} \quad (4.4)$$

For $k \in \{0, 1, \dots, 2047\}$, set $y = (2k + 1)/512$ and take the increasing list of real roots in x of

$$\begin{aligned} 0 = & -0.072399902677\psi_1(y) + 0.073767456734\psi_3(y) \\ & - 0.71483345535\psi_1(x) + 0.008648628127\psi_1(x)\psi_2(y) \\ & + 0.043678832526\psi_2(x)\psi_1(y) - 0.004614640469\psi_2(x)\psi_3(y) \\ & + 0.036990218101\psi_3(x) + 0.001978721807\psi_3(x)\psi_2(y) \\ & + 0.036193792294\psi_4(x)\psi_1(y) + 0.131004390897\psi_5(x) \\ & - 0.011093407361\psi_7(x). \end{aligned} \quad (4.5)$$

There are five roots for $0 \leq k \leq 1052$, three for $1053 \leq k \leq 1535$, and one for $1536 \leq k < 2048$. For each k , round these roots to the nearest multiples of 10^{-12} , then apply the numerical corrections specified in Appendix C. Denote the resulting numbers, in increasing order, by $r_{k,1}, r_{k,2}, \dots$. Define

$$F_*(x, y) := \begin{cases} 1 + 2 \sum_{a: r_{k,a} \leq x} (-1)^a, & k/256 \leq y < (k+1)/256, \quad 0 \leq k < 2048, \\ -\text{sign}(x), & y \geq 8, \\ -F_*(-x, -y), & y < 0. \end{cases}$$

The rounding algorithm. The input is a family of unit vectors $x_1, \dots, x_m, y_1, \dots, y_n$ as in Eq. (1.1); write $t = \langle x_a, y_b \rangle$ for a row-column pair.

1. Sample one index $i \in \{0, \dots, 31\}$ with probability λ_i , common to all row and column positions.
2. For $i \neq 14$, take inner products of the following vectors with one shared Gaussian vector whose coordinates are independent $\mathcal{N}(0, 1)$:

$$\begin{aligned} x_a & \mapsto \bigoplus_{m \geq 1} \sqrt{|[t^m]P_i(t)|} x_a^{\otimes m} \oplus \left(\sqrt{1 - \sum_{m \geq 1} |[t^m]P_i(t)|}, 0 \right), \\ y_b & \mapsto \bigoplus_{m \geq 1} \text{sign}([t^m]P_i(t)) \sqrt{|[t^m]P_i(t)|} y_b^{\otimes m} \oplus \left(0, \sqrt{1 - \sum_{m \geq 1} |[t^m]P_i(t)|} \right). \end{aligned}$$

The direct sums range over m with $[t^m]P_i \neq 0$. The images are unit vectors with cross inner product $P_i(t)$, since

$$\|x_a^{\otimes m}\| = \|y_b^{\otimes m}\| = 1, \quad \langle x_a^{\otimes m}, y_b^{\otimes m} \rangle = t^m, \quad \sum_{m \geq 1} |[t^m]P_i| + 1 - \sum_{m \geq 1} |[t^m]P_i| = 1.$$

3. For $i = 0, \dots, 13, 15, 16$, repeat step 2 with $-t$ in place of P_i and three further independent Gaussian vectors; apply F to the four coordinates at each position. For $i = 14$, use the displayed maps with t^3, t^5 in place of P_i and two independent Gaussian vectors; apply $F_*(x, y)$ on rows and $F_*(-x, y)$ on columns. For $17 \leq i \leq 31$, apply sign to the coordinate from step 2. Each Gaussian vector is shared across all positions, and the output is one sign per input vector.

Analysis. For each $i \neq 14$, on the graded Gaussian space of Section 3 put

$$W(X) := \sum_{m \geq 1} \sqrt{|[t^m]P_i|} X_4^{(m)} + \sqrt{1 - \sum_{m \geq 1} |[t^m]P_i|} X_5,$$

$$W'(X) := \sum_{m \geq 1} \text{sign}([t^m]P_i) \sqrt{|[t^m]P_i|} X_4^{(m)} + \sqrt{1 - \sum_{m \geq 1} |[t^m]P_i|} X_6.$$

Let μ choose, with probability λ_i , the pair

$$(f(X), g(X)) = \begin{cases} (F(W(X), X_1, X_2, X_3), F(W'(X), -X_1, -X_2, -X_3)), & i = 0, \dots, 13, 15, 16, \\ (F_*(X_1^{(3)}, X_1^{(5)}), F_*(-X_1^{(3)}, X_1^{(5)})), & i = 14, \\ (\text{sign } W(X), \text{sign } W'(X)), & 17 \leq i \leq 31. \end{cases}$$

All nonzero degrees of P_i are odd, and F, F_* are odd almost everywhere; thus these pairs are graded odd. Define

$$H_i := H_{f,g} \quad \text{for the pair indexed by } i, \quad H := H_\mu = \sum_{i=0}^{31} \lambda_i H_i, \quad c_m := [t^m]H = c_m(\mu).$$

For graded t -correlated (X, Y) ,

$$\mathbb{E}[W(X)W'(Y)] = P_i(t), \quad \mathbb{E}[X_j(-Y_j)] = -t \quad (1 \leq j \leq 3), \quad \mathbb{E}[X_1^{(m)}Y_1^{(m)}] = t^m \quad (m = 3, 5).$$

Different coordinate pairs are independent, and each coordinate is standard Gaussian. Consequently the algorithm satisfies

$$\mathbb{E}[\text{row sign at } x_a \cdot \text{column sign at } y_b \mid i] = H_i(\langle x_a, y_b \rangle).$$

By Eq. (3.3) and Lemmas 2.5 and 3.3,

$$\sum_{m \geq 0} |[t^m]H_i| \leq 1, \quad H_i(-t) = -H_i(t), \quad c(\mu) \in \text{conv}(\mathcal{B}) \subseteq \bar{\mathcal{C}}.$$

In particular, the series for H_i, H converge absolutely and uniformly on $|t| \leq 1$ and are holomorphic on $|t| < 1$.

Theorem 4.2. *For the rounding scheme above,*

$$c_1 - \sum_{\substack{m \geq 3 \\ m \text{ odd}}} |c_m| > 0.561831528950464991 > \frac{1000000}{1779893}.$$

In particular, $K_G < 1.779893$.

Proof. Since the even coefficients vanish, the bounds Eq. (C.5), Eq. (C.30), Eq. (C.38), Eq. (C.40) give

$$\begin{aligned}
c_1 - \sum_{m \geq 2} |c_m| &\geq c_1 - \sum_{m=2}^{681} |c_m| - \lambda_{14} \sum_{m > 681} |[t^m]H_{14}| - \sum_{i=17}^{31} \lambda_i \sum_{m > 681} |[t^m]H_i| \\
&\quad - \sum_{i=1}^{13} \lambda_i \sum_{m > 681} |[t^m]H_i| - \sum_{m > 681} \left| [t^m] \sum_{i \in \{0,15,16\}} \lambda_i H_i \right| \\
&> 0.5618773799453352 - 0.000003878150168295 \\
&\quad - 0.000034097111321496 - 0.000007875733380418 \\
&= 0.561831528950464991 > \frac{1000000}{1779893}.
\end{aligned}$$

Since $c(\mu) \in \bar{\mathcal{C}}$ and $c \mapsto c_1 - \sum_{m \geq 2} |c_m|$ is continuous in ℓ_1 , Eq. (2.2) yields

$$K_G \leq \left(c_1 - \sum_{m \geq 2} |c_m| \right)^{-1} < \frac{1779893}{1000000}.$$

□

Table 1: Weights and polynomial coefficients before normalization.

i	weight	t	t^3	t^5
0	0.002425733108742487	0.946	-0.054	0
1	0.00000701460381171968	0	-1	0
2	0.0000037499635434410743	0.05	-0.95	0
3	0.000009920220049838783	0.075000000000000001	-0.925	0
4	0.00006987843003609716	0.1	-0.9	0
5	0.00012479539456651385	0.2	-0.8	0
6	0.00004671038151698264	0.225	-0.775	0
7	0.0004249077174049799	0.35000000000000003	-0.6499999999999999	0
8	0.00018043956905586483	0.050170753056963675	-0.7540766094444311	0.19575263749860508
9	0.00000901084741869339	0.01741176476977869	-0.9229528872821551	0.05963534794806634
10	0.00010362576993944439	0.03393956633363427	-0.655490081870693	0.3105703517956727
11	0.00006378679383008387	-0.0036112846919679523	-0.9881992863232164	-0.008189428984815658
12	0.000050102169538197746	0.0009803141322194	-0.5234190673248801	0.4756006185429005
13	0.000013977989256200987	0.0008970753092140672	-0.4493368582298268	0.5497660664609592
14	0.000035687962816869	-	-	-
15	0.7470177879401771	0.9029200000000001	-0.09707999999999994	0
16	0.22290213190118147	0.90294	-0.09706000000000004	0

i	weight	t	m	t^m
17	0.026477721663446113	0.8859999999999999	3	-0.11400000000000001
18	0.00000665613397460388	0.96	3	-0.040000000000000036
19	0.0000009997984826272776	0.1	13	-0.9
20	0.000000390115436625487	0.525	13	-0.475
21	0.0000012374573661838554	0.5499999999999999	13	-0.45000000000000007
22	0.0000011172456644699994	0.7	13	-0.30000000000000004
23	0.0000004608585343191249	0.725	13	-0.275
24	0.000004808038589224191	0.7999999999999999	13	-0.20000000000000007
25	0.00005442381630242658	0.9249999999999999	13	-0.07500000000000007
26	0.000008538350059191554	0.95	13	-0.050000000000000044
27	0.000000336030804208466	0.1	15	0.9
28	0.00000030196245224993114	0.375	15	0.625
29	0.0000012754666931590235	0.4	15	0.6
30	0.000005988121471482638	0.6749999999999999	15	0.32500000000000007
31	0.000003892292284216072	0.7	15	0.30000000000000004

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A Numerical summary of the primal/dual run

We report the statistics used for one representative execution of the primal/dual search. We have attempted to use more reproducible methods for this run instead of pushing for the absolute tightest bounds on K_G .

Runtime statistics. The maximum degree parameter was $D = 15$. The initialization was $Q = \Pi_1$. There were 92 update iterations for Q . The final response pool had $|\mathcal{R}| = 274$ rounding function pairs (f, g) . A pair was added to $\mathcal{R}$ if they significantly exceeded the current maximum value of Q on $\mathcal{R}$. A larger pool of 13,069 asymptotic Boolean functions (see Section 3) was maintained to heuristically estimate the ∞ -to-1 norm of Q . The search took 190 minutes using GPT-6 Astra. After 60 minutes, the algorithm began iteration 82. After 115 minutes, the algorithm began iteration 90. The remaining 75 minutes were spent searching for ∞ -to-1 optimizers and updating Q only two more times.

The final candidate μ was computed after the iteration using Eq. (2.6). The LP was equivalent to maximizing $c_1(\mu)$ subject to $c_3 = c_5 = \dots = c_{15} = 0$ (a useful variant would be to increase D in this calculation in order to minimize more low-degree coefficients of μ , but in this run we kept $D = 15$). Furthermore, in order to measure the progress of upper bounds throughout the iteration, we additionally ran Eq. (2.6) at each iteration. The change in the heuristic upper and lower bounds is plotted in Fig. 1.

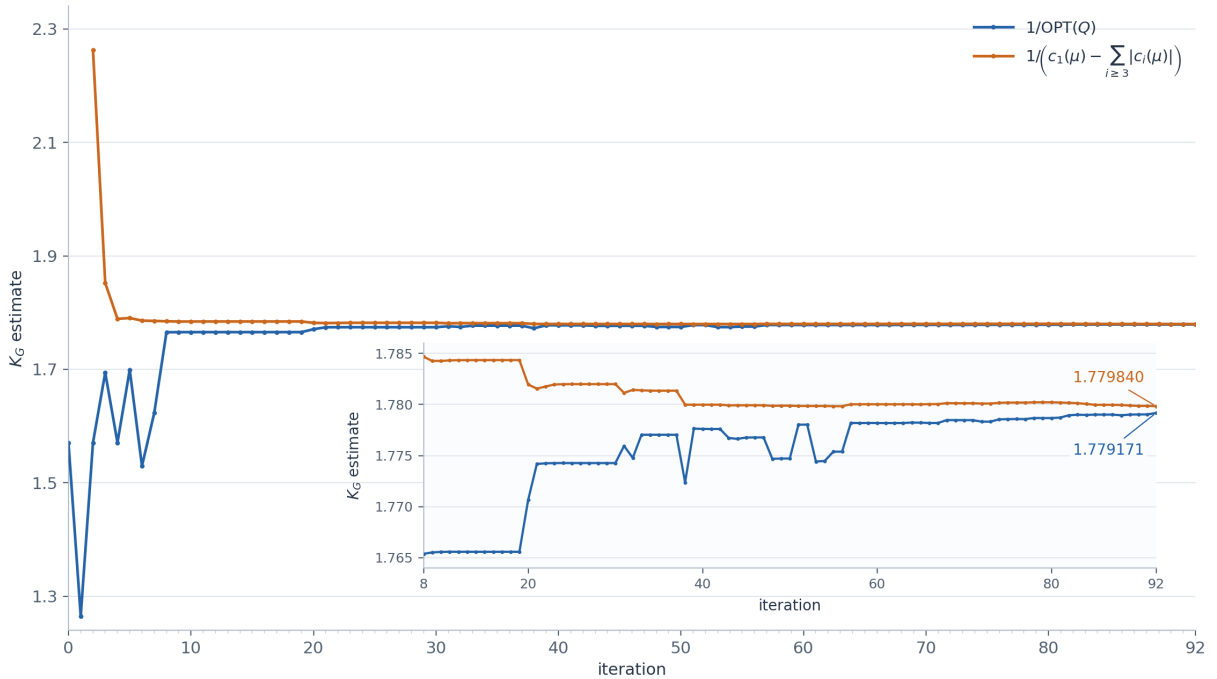


Figure 1: Primal/dual estimates of K_G by iteration. The value $\text{OPT}(Q^{(t)})$ is estimated using the final pool $\mathcal{R}$ by the formula $\text{OPT}(Q^{(t)}) \approx \max_{(f,g) \in \mathcal{R}} \langle f, Q^{(t)} g \rangle$. The estimate of $\sum_{i \geq 3} |c_i(\mu)|$ is only computed up to degree at least 63. At the end of the iteration, sources contributing to the remaining gap include a nonzero duality gap, only optimizing cancellations in μ up to degree 15, and numerical integration error.

Outputs. The final Hermite projection game has degree 15 with $Q = \sum_{\text{odd } 1 \leq m \leq 15} q_m \Pi_m$ where the coefficients q_m are in Table 2.

m	q_m	m	q_m
1	1	9	0.13649017557899745
3	-0.879274051265278	11	-0.049432632669210025
5	0.5207487584184273	13	-0.10368806013060777
7	-0.3110788699180306	15	0.12636021488829785

Table 2: Coefficients for the candidate degree-15 Hermite game produced by the run.

During the final iteration, when Q was selected to minimize the value of the pool via Eq. (2.5), the maximum value among the pool functions was $\max_{(f,g) \in \mathcal{R}} \langle f, Qg \rangle = 0.5619674$. A subsequent search for new responses to Q found functions with slightly larger value,

$$\max_{(f,g) \in \mathcal{R}} \langle f, Qg \rangle = 0.5620595, \quad \frac{1}{0.5620595} = 1.779171 \dots$$

Thus, the duality gap at termination was still nonzero. The ∞ -to-1 calculations were getting long enough that it was very expensive to continue the iteration.

The final mixture of rounding functions μ is supported on 8 pairs of rounding functions (f, g) each with the form $\text{sign}(p(X))$ for a graded multivariate polynomial p with graded-degree at most 15. The mixture weights are given in Table 3. All 8 rounding pairs have value approximately 0.5619674 in the final game Q , the maximum among functions in the pool (as expected by complementary slackness). The numerical estimates are $c_1(\mu) \approx 0.5619674$ and $\sum_{\text{odd } i=3}^{63} |c_i(\mu)| \approx 0.0001118$. The resulting estimated upper bound is

$$c_1(\mu) - \sum_{i=3}^{63} |c_i(\mu)| \approx 0.5618556, \quad \frac{1}{0.5618556} = 1.7798167 \dots$$

i	w_i	Function family
1	0.07659708908805163	Graded linear threshold
2	0.38070941881309583	Graded linear threshold
3	0.3422214571899304	Graph threshold, $n = 8$
4	0.027934013874205684	Bivariate threshold, $n = 2$
5	0.04989420667246783	Bivariate threshold, $n = 2$
6	0.05838336532409958	Symmetric polynomial threshold, $n = 32$
7	0.017645044351596674	Symmetric polynomial threshold, $n = 32$
8	0.04661540468655233	Symmetric polynomial threshold, $n = 8$

Table 3: Weights and descriptions of the final rounding mixture. n is the number of graded variables used in the function. Full descriptions are available at <https://github.com/christophsjones/RealGrothendieck>.

B Lower bound certificate

B.1 The game and its one-variable reduction

In this Section, we prove Theorem 4.1. For measurable $f, g: \mathbb{R}^n \rightarrow [-1, 1]$, write

$$Q = Q^{(n)} = \sum_m q_m \Pi_m, \quad v_n(q) = \text{OPT}(Q) = \sup_{f,g} |\langle f, Qg \rangle|. \quad (\text{B.1})$$

By a standard discretization argument to approximate Q by finite matrices [Pis12, Theorems 2.4 and 2.5], we have $K_G \geq \frac{1}{v_\infty(q)}$. It remains to upper bound $v_\infty(q)$.

Define coefficients q_m from Eq. (4.1). For $N = 7, 9$, choose rational weights $w^{(N)} \in \mathbb{R}^{(N-1)/2}$ and put

$$B_N := \max_{\sigma \in \{-1,1\}^N} \frac{1}{N} \sum_{r=0}^{N-1} \sum_{j=1}^{(N-1)/2} w_j^{(N)} \sigma_r \sigma_{r+j}, \quad \sigma_{r+N} = -\sigma_r, \quad (\text{B.2})$$

$$\Gamma_m := \sum_{N \in \{7,9\}} \sum_{j=1}^{(N-1)/2} w_j^{(N)} \cos^m(\pi j/N), \quad B = B_7 + B_9. \quad (\text{B.3})$$

Global sign reversal leaves the objective unchanged, so one may fix $\sigma_0 = 1$. Define, with all other entries zero,

$$\alpha_m := q_m - \Gamma_m \quad (m = 3, 5, \dots, 21), \quad \beta_m := -q_m - \Gamma_m \quad (m = 1, 3, \dots, 21). \quad (\text{B.4})$$

Set $\psi := (\psi_0, \dots, \psi_{21})^T$ and call $M := \begin{pmatrix} \mathbb{E}[\psi u] \\ \mathbb{E}[\psi v] \end{pmatrix} \in \mathbb{R}^{44}$ **feasible** when $u, v: \mathbb{R} \rightarrow [-1, 1]$ are measurable. Index its coordinates by $0, \dots, 43$ and set, for $p = 0, 1$ and $0 \leq j \leq 10$,

$$(h_p(M))_j := \frac{M_{2j+p} + M_{22+2j+p}}{2}, \quad (k_p(M))_j := \frac{M_{2j+p} - M_{22+2j+p}}{2}. \quad (\text{B.5})$$

Choose symmetric rational 11×11 matrices H_p, K_p , vectors $\ell_h, \ell_k \in \mathbb{Q}^{11}$, and rational t, c_h, c_k, C . Set $\varepsilon := 10^{-7}$ and $D_{p,r}(a) := \text{diag}(a_{2j+p+r})_{j=0}^{10}$. Require

$$\Gamma_m \geq 0 \quad (m > 21 \text{ odd}), \quad 1 - \Gamma_1 > 0. \quad (\text{B.6})$$

For each $(A, a, \lambda, c) \in \{(H, \alpha, \ell_h, c_h), (K, \beta, \ell_k, c_k)\}$, assume, $\forall p = 0, 1$ and $1 \leq r \leq 21$,

$$A_p \succeq 0, \quad A_p + \varepsilon I_{11} \succeq D_{p,r}(a), \quad \begin{pmatrix} A_1 + \varepsilon I_{11} - D_{1,0}(a) & \lambda \\ \lambda^T & c + \varepsilon \end{pmatrix} \succeq 0. \quad (\text{B.7})$$

Here $\succeq 0$ means positive semidefinite, and I_{11} is the 11×11 identity matrix. Unsubscripted norms of finite vectors are Euclidean; $\|z\|_1 = \sum_j |z_j|$. Define

$$F(M) := \sum_{p=0}^1 (h_p(M)^T H_p h_p(M) + k_p(M)^T K_p k_p(M)) + t(h_1(M))_0 - 2\ell_h^T h_1(M) - 2\ell_k^T k_1(M), \quad (\text{B.8})$$

$$\Phi(x) := B + c_h + c_k + 3\varepsilon + (1 - \Gamma_1) \frac{2}{\pi} x^2 - t \sqrt{2/\pi} x. \quad (\text{B.9})$$

Write $F(M) = M^T R M + \ell^T M$ with R symmetric. By the invertible change of variables Eq. (B.5), $R \succeq 0$ is equivalent to $H_p, K_p \succeq 0$ for $p = 0, 1$.

Proposition B.1 (One-variable criterion). *Suppose a finite family of closed intervals I covers $[0, 1]$, each with dimension-independent parameters satisfying Eq. (B.6), Eq. (B.7), and*

$$F(M) \leq C \quad \text{for every feasible } M, \quad (\text{B.10})$$

$$C + \max_{x \in \partial I} \Phi(x) \leq v_* = \frac{1000}{1773} - 10^{-9}. \quad (\text{B.11})$$

Here ∂I denotes the interval endpoints. Then $v_n(q) \leq v_*$ for every $n \geq 1$, hence Theorem 4.1.

Proof. Since Q is self-adjoint with odd range, successive replacements of $f = \text{sign}(Qg)$ and $g = \text{sign}(Qf)$ cannot decrease the OPT value. Use $\text{sign}(z_1)$ wherever the argument vanishes; both responses are then odd almost everywhere. Put $h := (f + g)/2$, $k := (f - g)/2$; then $h^2 + k^2 = 1$.

Rotate the domain so that $\Pi_1 h(z)$ is a nonnegative multiple of z_1 . Then $\|\Pi_1 h\|_2 = \mathbb{E}[z_1 h] \leq \mathbb{E}|z_1| = \sqrt{2/\pi}$, so $x := \sqrt{\pi/2}\|\Pi_1 h\|_2 \in [0, 1]$ and $\Pi_1 h = \sqrt{2/\pi} x z_1$. Choose an interval $I \ni x$ and use its associated parameters.

For each N and independent $U, V \sim \gamma_n$, set $\sigma_r := f(\cos(r\pi/N)U + \sin(r\pi/N)V)$. Oddness gives $\sigma_{r+N} = -\sigma_r$. The correlation identity Eq. (3.3) gives

$$\sum_j w_j^{(N)} \sum_{m \text{ odd}} \cos^m(\pi j/N) \|\Pi_m f\|_2^2 = \mathbb{E} \left[\frac{1}{N} \sum_{r=0}^{N-1} \sum_j w_j^{(N)} \sigma_r \sigma_{r+j} \right] \stackrel{\text{Eq. (B.2)}}{\leq} \mathbb{E}[B_N] = B_N. \quad (\text{B.12})$$

All series converge absolutely by Parseval. Average for f, g and sum over N , using $(\|\Pi_m(f)\|_2^2 + \|\Pi_m(g)\|_2^2)/2 = \|\Pi_m h\|_2^2 + \|\Pi_m k\|_2^2$. Combining this with $\langle f, Qg \rangle = \sum_m q_m (\|\Pi_m h\|_2^2 - \|\Pi_m k\|_2^2)$ and $q_1 = 1$, $q_m = 0$ for $m > 21$ from Eq. (4.1), gives

$$\begin{aligned} \langle f, Qg \rangle &= \sum_{m \text{ odd}} \Gamma_m (\|\Pi_m h\|_2^2 + \|\Pi_m k\|_2^2) + \sum_{m \text{ odd}} ((q_m - \Gamma_m) \|\Pi_m h\|_2^2 + (-q_m - \Gamma_m) \|\Pi_m k\|_2^2) \\ &\stackrel{\text{Eq. (B.12)} \wedge \text{Eq. (B.3)} \wedge \text{Eq. (B.4)} \wedge \text{Eq. (B.6)}}{\leq} B + (1 - \Gamma_1) \frac{2}{\pi} x^2 + \sum_m (\alpha_m \|\Pi_m h\|_2^2 + \beta_m \|\Pi_m k\|_2^2). \end{aligned} \quad (\text{B.13})$$

Write the Gaussian coordinates as independent $S \sim \gamma_1$, $Y \sim \gamma_{n-1}$, with Y the empty tuple if $n = 1$. Let $M(Y)$ be the feasible moments of $f(\cdot, Y), g(\cdot, Y)$. For $p = 0, 1$ and $\eta \in \{0, 1, \dots\}^{n-1}$, set

$$\begin{aligned} u_p(Y) &:= h_p(M(Y)), & v_p(Y) &:= k_p(M(Y)), \\ \hat{u}_p(\eta) &:= \mathbb{E}[u_p(Y) \psi_\eta(Y)], & \hat{v}_p(\eta) &:= \mathbb{E}[v_p(Y) \psi_\eta(Y)]. \end{aligned}$$

The last sum in Eq. (B.13) equals

$$\sum_{\eta} \sum_{p=0}^1 [\hat{u}_p(\eta)^T D_{p,|\eta|}(\alpha) \hat{u}_p(\eta) + \hat{v}_p(\eta)^T D_{p,|\eta|}(\beta) \hat{v}_p(\eta)].$$

Also $(h_1(M(Y)))_0 = (M_1(Y) + M_{23}(Y))/2 = \mathbb{E}_S[Sh(S, Y)]$, so $\mathbb{E}_Y(h_1(M(Y)))_0 = \sqrt{2/\pi} x$. In the Hermite expansion in Y , coordinate j of $h_p(M), k_p(M)$ at index η has total degree $2j + p + |\eta|$ in (S, Y) . For $1 \leq |\eta| \leq 21$, apply $D_{p,|\eta|}(\alpha) \preceq H_p + \varepsilon I_{11}$ and $D_{p,|\eta|}(\beta) \preceq K_p + \varepsilon I_{11}$ from Eq. (B.7). For $|\eta| > 21$, both diagonal matrices vanish and $H_p, K_p \succeq 0$ by the same equation. When $\eta = 0$ and $p = 0$, $D_{0,0}(\alpha) = D_{0,0}(\beta) = 0$. For $\eta = 0$ and $p = 1$, the last inequality of Eq. (B.7), applied to $(z, -1)$, gives the following bounds for $z = \hat{u}_1(0)$ and $z = \hat{v}_1(0)$, respectively:

$$z^T D_{1,0}(\alpha) z \leq z^T H_1 z - 2\ell_h^T z + c_h + \varepsilon + \varepsilon \|z\|_2^2, \quad z^T D_{1,0}(\beta) z \leq z^T K_1 z - 2\ell_k^T z + c_k + \varepsilon + \varepsilon \|z\|_2^2.$$

Parseval adds these bounds. The constants cost $c_h + c_k + 2\varepsilon$, and the quadratic errors cost at most

$$\varepsilon \mathbb{E}_Y \sum_p (\|h_p(M(Y))\|^2 + \|k_p(M(Y))\|^2) \leq \varepsilon \mathbb{E}(h^2 + k^2) = \varepsilon$$

by Bessel's inequality. Combining these estimates into Eq. (B.13) then gives

$$\langle f, Qg \rangle \leq \mathbb{E}_Y F(M(Y)) + \Phi(x) \stackrel{\text{Eq. (B.10)}}{\leq} C + \Phi(x). \quad (\text{B.14})$$

By Eq. (B.6), Φ is convex. Since $x \in I$, $C + \Phi(x) \leq C + \max_{\partial I} \Phi \leq v_*$ by Eq. (B.11), proving the claim. $\square$

B.2 Finite certification

Since $R \succeq 0$, we reduce $F \leq C$ on feasible moments to an integer quadratic in signs.

Reduction to finitely many signs. Partition $[-9, 9]$ into $J = [j/2, (j+1)/2]$, $-18 \leq j \leq 17$, and divide each J into n_J panels of length $h_J = 1/(2n_J)$. On a panel $[a, b]$, put $\tau = (s-a)/(b-a)$ and

$$(\lambda_0, \lambda_1, \lambda_2) := ((1-\tau)^2, 2\tau(1-\tau), \tau^2), \quad (c_0(p), c_1(p), c_2(p)) := \left(p(a), 2p\left(\frac{a+b}{2}\right) - \frac{p(a)+p(b)}{2}, p(b) \right).$$

The interpolant is $I_2p = \sum_\nu \lambda_\nu c_\nu(p)$, with $\lambda_\nu \geq 0$. Form V from the columns $c_\nu(\psi) \int_a^b \lambda_\nu d\gamma_1$, merging those at shared endpoints (positive multiples of the same $\psi(s)$) and ordering by associated points. Set $\mathcal{B} = \text{diag}(V, V)$; it has 44 rows and $N_{\text{sign}} = 4 \sum_J n_J + 2$ columns.

For feasible M , set $\xi = 2RM + \ell$, $p_i = \psi^T \xi^{[i]}$, and $\sigma_j = \text{sign}((\mathcal{B}^T \xi)_j)$; brackets indicate 22-coordinate blocks, or 22×22 matrix blocks. Convexity, $\lambda_\nu \geq 0$, and the error bound Eq. (B.17) below give

$$F(M) - F(\mathcal{B}\sigma) \leq \xi^T M - \|\mathcal{B}^T \xi\|_1 \leq \sum_i \int |p_i| d\gamma_1 - \|\mathcal{B}^T \xi\|_1 \leq E. \quad (\text{B.15})$$

The analytic error. Quadratic interpolation satisfies, for $p \in C^3([0, 1])$,

$$\int_0^1 |p - I_2p| ds \leq \frac{1}{96} \int_0^1 |p'''| ds. \quad (\text{B.16})$$

Indeed, choose $g''' = |p'''|$ and put $w(s) = \text{sign}(1/2 - s)$. The Lagrange remainder for $g \pm p$ gives $|p - I_2p| \leq w(g - I_2g)$. For $H(s) = u^2(3 - 4u)/24$, $u = \min(s, 1 - s)$, we have $H \in C^2$, $H = H' = 0$ at the endpoints, $-H''' = w$ almost everywhere, and $0 \leq H \leq 1/96$. Three integrations by parts against $g - I_2g$, which vanishes at the endpoints, bound the left side by $\int_0^1 H |p'''|$, proving the claim. On a panel in J , scaling contributes h_J^3 and comparison of Gaussian densities contributes at most $e^{h_J \max_J |s|}$.

Let $T = \{|s| > 9\}$ and let $\mathcal{P}_0$ partition $\mathbb{R}$ into the 36 intervals J and two exterior intervals, up to endpoints. Put

$$G_L = \int_L \psi \psi^T d\gamma_1, \quad Q_J = \int_J \psi''' (\psi''')^T d\gamma_1,$$

and, for $Q \in \mathbb{R}^{22 \times 22}$, $Q \succeq 0$, and a finite measurable partition $\mathcal{P}$, define

$$\mathcal{A}(Q; \mathcal{P}) = \sum_{i=1}^2 \sqrt{(\ell^{[i]})^T Q \ell^{[i]}} + 2 \sum_{i,j=1}^2 \sum_{L \in \mathcal{P}} \sqrt{\gamma_1(L) \text{Tr}((R^{[ij]})^T Q R^{[ij]} G_L)}.$$

Splitting each feasible moment over $\mathcal{P}$ and applying the triangle inequality gives $\sum_i \|Q^{1/2} \xi^{[i]}\| \leq \mathcal{A}(Q; \mathcal{P})$, since Cauchy–Schwarz gives, for a real matrix R_0 with 22 columns and measurable $h: \mathbb{R} \rightarrow [-1, 1]$,

$$\left\| R_0 \int_L \psi h d\gamma_1 \right\|^2 \leq \gamma_1(L) \text{Tr}(R_0 G_L R_0^T) \quad (|h| \leq 1).$$

Another Cauchy–Schwarz inequality bounds the derivative and tail integrals. Thus the uniform error allowance is any rational E with

$$E \geq \sqrt{\gamma_1(T)} \mathcal{A}(G_T; \{\mathbb{R}\}) + \sum_J \frac{e^{h_J \max_J |s|} h_J^3}{96} \sqrt{\gamma_1(J)} \mathcal{A}(Q_J; \mathcal{P}_0), \quad (\text{B.17})$$

where $G_{\mathbb{R}} = I_{22}$. The polynomial integrands have degree at most 42; their Gaussian moments follow from Eq. (C.58), using reflection and infinite-endpoint limits for the tails. The verifier uses equivalent Hermite identities.

Integer rounding and the target. Bars denote rounding $\mathcal{B}$ to multiples of 2^{-42} and R, ℓ to multiples of 2^{-40} , with $\bar{R}$ symmetric. With entrywise absolute values, set $b_j = \sum_k |\bar{\mathcal{B}}_{jk}|$, choose rational $\delta_j \geq \sum_k |\mathcal{B}_{jk} - \bar{\mathcal{B}}_{jk}|$, and put $b_+ = b + \delta$. The bounds $|\bar{\mathcal{B}}\sigma| \leq b$ and $|\mathcal{B}\sigma - \bar{\mathcal{B}}\sigma| \leq \delta$ give, by expansion, the uniform input-error allowance

$$E_{\text{input}} \geq \delta^T |\bar{R}|(2b + \delta) + b_+^T |R - \bar{R}|b_+ + |\bar{\ell}|^T \delta + |\ell - \bar{\ell}|^T b_+. \quad (\text{B.18})$$

After substituting $\sigma_j^2 = 1$, multiply the rounded quadratic by 2^{44} and round its coefficients to nearest integers, obtaining $\bar{P} = a_0 + \sum_j a_j \sigma_j + \sum_{j < k} a_{jk} \sigma_j \sigma_k$. Sign monomials have absolute value one, so the sum of absolute coefficient-rounding errors bounds the uniform error. Let E_{graph} be this sum divided by 2^{44} , including the once-rounded total diagonal. Thus, with $\Delta = E_{\text{input}} + E_{\text{graph}}$, $|F(\mathcal{B}\sigma) - 2^{-44}\bar{P}(\sigma)| \leq \Delta$. Equation (B.15) gives $F(M) \leq E + \Delta + 2^{-44}\bar{P}(\sigma)$, so it suffices to certify

$$\max_{\sigma \in \{-1, 1\}^{N_{\text{sign}}}} \bar{P}(\sigma) \leq U := \lfloor 2^{44}(C - E - \Delta) \rfloor. \quad (\text{B.19})$$

Graph bounds and sign fixing. Use vertices j^+, j^- and terminals of signs $+1, -1$. For each nonzero a_{jk} , join equal copies if $a_{jk} > 0$ and opposite copies otherwise, with capacity $|a_{jk}|$ on both edges. Join j^+ to terminal $\text{sign}(a_j)$ and j^- to the opposite terminal, with capacity $|a_j|$; omit zero capacities. A cut assigns vertex signs and sums capacities across opposite signs. For consistent cuts, the signs are $(\sigma_j, -\sigma_j)$, giving

$$\max_{\sigma} \bar{P}(\sigma) \leq a_0 + \sum_j |a_j| + \sum_{j < k} |a_{jk}| - \min_x \text{cap}(x), \quad (\text{B.20})$$

with equality when minimizing over consistent cuts. Represent each edge by opposite directed arcs. Check flow nonnegativity, capacity bounds, and conservation at nonterminals. Summing conservation shows that a flow and cut of equal value certify a minimum cut.

A minimum cut x also fixes opposite pairs. Replacing each $x_{j\pm}$ by $-x_{j\mp}$ preserves capacity. For consistent z , sort z, x , and this reflected cut coordinatewise into $a \leq y \leq b$. Sorting cannot increase total capacity: each edge's disagreements become the absolute difference of its endpoint counts of positive signs. By minimality,

$$\text{cap}(y) \leq \text{cap}(z) + 2\text{cap}(x) - \text{cap}(a) - \text{cap}(b) \leq \text{cap}(z).$$

The coordinatewise median satisfies $y_{j+} = -y_{j-}$ and $y_{j\pm} = x_{j\pm}$ whenever $x_{j+} = -x_{j-}$. Thus some maximizing assignment obeys all fixings. Substitute them, retaining constants, and split remaining signs as needed. Check both children and require every terminal bound to be at most U .

Rectangle certificates. Checked invariance of P under grid reflection combined with negation of both sign blocks allows the first block's sign at zero to be fixed: $\sigma_{j_0} = 1$. For supplied nonzero $v_1, v_2 \in \mathbb{Z}^{N_{\text{sign}}}$, put $u_k = v_k^T \sigma / \|v_k\|_1 \in [-1, 1]$. Recursively split $[-1, 1]^2$ at rational interior points, retaining both closed children. On a leaf $\mathcal{R} = \prod_k [l_k, h_k]$, add

$$2^{44} \sum_{k=1}^2 \mu_k (u_k - l_k)(h_k - u_k), \quad \mu_k \geq 0. \quad (\text{B.21})$$

The correction is nonnegative on $\mathcal{R}$. Expand with $\sigma_j^2 = 1$ and round each coefficient contribution, including the separate constants $-2^{44}\mu_k l_k h_k$ and $-2^{44}\mu_k \|v_k\|^2 / \|v_k\|_1^2$, to a nearest integer. For each k , charge half the number of nonintegral contributions, rounded upward, to cover their absolute rounding

errors. Apply graph bounds and sign fixing to the augmented polynomial over all signs with $\sigma_{j_0} = 1$, unrestricted by $\mathcal{R}$. Its bound plus the charges must be at most U .

Alternatively, supplied integers a, b exclude a leaf if, for $w = av_1 + bv_2$,

$$\min_{(s_1, s_2) \in \mathcal{R}} (a\|v_1\|_1 s_1 + b\|v_2\|_1 s_2) > w_{j_0} + \sum_{j \neq j_0} |w_j|.$$

The left side is evaluated at the minimizing corner; the right side is $\max_{\sigma: \sigma_{j_0}=1} w^T \sigma$. The complete cover and checked leaves prove [Eq. \(B.19\)](#).

Acceptance and reproduction. The infinite part of [Eq. \(B.6\)](#) follows from the rational inequalities

$$\eta_N := w_1^{(N)} - \sum_{j=2}^{(N-1)/2} \max\{0, -w_j^{(N)}\} > 0 \quad (N = 7, 9). \quad (\text{B.22})$$

Indeed, for $m \geq 0$ and $c_j = \cos(\pi j/N)$, $\sum_j w_j^{(N)} c_j^m \geq c_1^m \eta_N > 0$, since $0 < c_j/c_1 \leq 1$.

The exact rows and subdivisions are in `data/lower_rows.json`; proof trees and verifier are in `lower/certificates/` and `lower/verify.py`, respectively. The supplied checks verify coverage of $[0, 1]$, rotation maxima, [Eq. \(B.22\)](#), $1 - \Gamma_1 > 0$, all reduction matrices, the unshifted $R \succ 0$, regenerated error allowances and integer inputs, the finite certificates, and [Eq. \(B.11\)](#). Analytic calculations use outward-rounded 384-bit intervals and positive LDL^T pivots for matrix inequalities; `lower/check-tail.py` checks [Eq. \(B.22\)](#) with exact rational arithmetic. All 35 rows pass, with largest payoff bound $0.564015352833 < v_*$, rounded upward. Hence [Eq. \(B.15\)](#) and [Eq. \(B.19\)](#) give $F \leq C$; [Proposition B.1](#) and [Theorem 2.2](#) prove [Theorem 4.1](#).

Install the README's dependencies, and run from the repository root:

```
python verify.py --lower --workers 4
```

This command includes the exact tail check and writes `runs/complete-*/ACCEPTED.json` only on success; `-data-only` checks input identities and structure, not the proof.

C Upper bound certificate

For the Hermite integrals below, W, X_1, X_2, X_3 are independent standard Gaussians and $X = (X_1, X_2, X_3)$. Define

$$C[a, b] := \sum_{|\alpha|=b} (\mathbb{E}[F(W, X) \psi_a(W) \psi_\alpha(X)])^2 \quad (a, b \geq 0, \alpha \in \{0, 1, \dots\}^3). \quad (\text{C.1})$$

The one-coordinate case of [Eq. \(3.3\)](#) and $\psi_a(-x) = (-1)^a \psi_a(x)$ give

$$H_i(t) = \sum_{a, b \geq 0} C[a, b] P_i(t)^a (-t)^b \quad (i = 0, \dots, 13, 15, 16), \quad (\text{C.2})$$

$$H_{14}(t) = \sum_{a, b \geq 0} (-1)^a (\mathbb{E}[F_*(X_1, X_2) \psi_a(X_1) \psi_b(X_2)])^2 t^{3a+5b}. \quad (\text{C.3})$$

For $17 \leq i \leq 31$, rotational symmetry of two independent standard Gaussians gives the sign-disagreement probability $\arccos(P_i(t))/\pi$, hence

$$H_i(t) = 1 - \frac{2}{\pi} \arccos P_i(t) = \frac{2}{\pi} \arcsin P_i(t) \quad (17 \leq i \leq 31). \quad (\text{C.4})$$

Equation [\(3.2\)](#) and Parseval give

$$C[a, b] \geq 0, \quad \sum_{a, b} C[a, b] = \mathbb{E} F^2 = 1, \quad C[a, b] = 0 \quad (a + b \text{ even}).$$

C.1 Numerical bounds for the coefficients up to degree 681

Lemma C.1. *The coefficients of H satisfy*

$$c_1 - \sum_{m=2}^{681} |c_m| > 0.5618773799453352. \quad (\text{C.5})$$

Proof. Define

$$F_{\text{tr}}(W, X) := F(W, X) \prod_{j=1}^3 \mathbf{1}_{\{|X_j| \leq 73/10\}}, \quad C_{\text{tr}}[a, b] := \sum_{|\alpha|=b} (\mathbb{E}[F_{\text{tr}}(W, X) \psi_a(W) \psi_\alpha(X)])^2,$$

and $\text{erfc}(x) := 2\pi^{-1/2} \int_x^\infty e^{-t^2} dt$. Choose $\hat{C}[a, b]$ as in Claim 1. By Parseval and Eq. (C.7),

$$\begin{aligned} \sum_{a, b \geq 0} |C_{\text{tr}}[a, b] - C[a, b]| &\leq \|F - F_{\text{tr}}\|_2 (\|F\|_2 + \|F_{\text{tr}}\|_2) \\ &\leq 2\sqrt{3 \text{erfc}(73/(10\sqrt{2}))} < 1.858282199124 \cdot 10^{-6}, \end{aligned} \quad (\text{C.6})$$

since $\|F_{\text{tr}}\|_2 \leq \|F\|_2 = 1$ and

$$\|F - F_{\text{tr}}\|_2^2 = \Pr\left(\max_j |X_j| > 73/10\right) \leq \sum_{j=1}^3 \Pr(|X_j| > 73/10) = 3 \text{erfc}(73/(10\sqrt{2})).$$

For square-summable real vectors x, y ,

$$\begin{aligned} \sum_j |x_j^2 - y_j^2| &\leq \|x - y\|_2 \|x + y\|_2 \leq \|x - y\|_2 (\|x\|_2 + \|y\|_2) \\ &\leq \|x - y\|_2 (2\|x\|_2 + \|x - y\|_2). \end{aligned} \quad (\text{C.7})$$

For $i \neq 14$, set $P_{i,+}(t) := \sum_{m \geq 1} [t^m] P_i(t) t^m$. Then

$$\begin{aligned} |[t^m](P_i(t)^a (-t)^b)| &\leq [t^m](P_{i,+}(t)^a t^b), \\ \sum_{m \geq 0} [t^m](P_{i,+}(t)^a t^b) &= P_{i,+}(1)^a \leq 1, \quad [t^m](P_{i,+}(t)^a t^b) = 0 \quad (m < a + b). \end{aligned} \quad (\text{C.8})$$

Only $a + b \leq 681$ contributes through degree 681, by Eq. (C.8). That equation, Eq. (C.10), and Eq. (C.6) give

$$\begin{aligned} &\sum_{m=1}^{681} \left| [t^m] \sum_{i \in \{0, \dots, 13, 15, 16\}} \lambda_i \sum_{a+b \leq 681} (\hat{C}[a, b] - C[a, b]) P_i(t)^a (-t)^b \right| \\ &\leq \left(\sum_{i \in \{0, \dots, 13, 15, 16\}} \lambda_i \right) \sum_{a+b \leq 681} |\hat{C}[a, b] - C[a, b]| \\ &< \frac{1315400222548137891645}{1351351351351351364726} (6.078879564946870 \cdot 10^{-8} + 1.858282199124 \cdot 10^{-6}) \\ &< 1.868016346073 \cdot 10^{-6}, \end{aligned} \quad (\text{C.9})$$

where the fraction is the sum of the indicated weights. For Eq. (C.8), induction from $P_i^0 = P_{i,+}^0 = 1$ uses

$$|[t^m] P_i^a| \leq \sum_{j=0}^m |[t^j] P_i| |[t^{m-j}] P_i^{a-1}| \leq [t^m] P_{i,+}^a.$$

Since $P_{i,+}(0) = 0$,

$$[t^m]P_{i,+}^a = 0 \quad (m < a), \quad [t^m](P_{i,+}^a t^b) = \begin{cases} [t^{m-b}]P_{i,+}^a, & m \geq b, \\ 0, & m < b. \end{cases}$$

The coefficient sum equals $P_{i,+}(1)^a \leq 1$. The triangle inequality and [Eq. \(C.11\)](#) yield

$$\begin{aligned} c_1 - \sum_{m=2}^{681} |c_m| &> 0.56187924796168130219 - 1.868016346073 \cdot 10^{-6} \\ &= 0.56187737994533522919 > 0.5618773799453352. \end{aligned}$$

□

Claim 1. *There exist nonnegative rational numbers $\hat{C}[a, b]$, $a + b \leq 681$, with*

$$\sum_{a+b \leq 681} |\hat{C}[a, b] - C_{\text{tr}}[a, b]| \leq 6.078879564946870 \cdot 10^{-8} \quad (\text{C.10})$$

and

$$\begin{aligned} [t] &\left(\sum_{i \in \{0, \dots, 13, 15, 16\}} \lambda_i \sum_{a+b \leq 681} \hat{C}[a, b] P_i(t)^a (-t)^b + \lambda_{14} H_{14}(t) + \sum_{i=17}^{31} \lambda_i H_i(t) \right) \\ &- \sum_{m=2}^{681} \left| [t^m] \left(\sum_{i \in \{0, \dots, 13, 15, 16\}} \lambda_i \sum_{a+b \leq 681} \hat{C}[a, b] P_i(t)^a (-t)^b + \lambda_{14} H_{14}(t) + \sum_{i=17}^{31} \lambda_i H_i(t) \right) \right| \quad (\text{C.11}) \\ &> 0.56187924796168130219. \end{aligned}$$

Proof. 1. Express the Hermite coefficients as integrals. Set $\phi(z) := e^{-z^2/2}/\sqrt{2\pi}$ and, for real s ,

$$B_j(s) := i^{-j} \mathbb{E}[e^{isr(X_1)} \psi_j(X_1) \mathbf{1}_{\{|X_1| \leq 73/10\}}], \quad B(s) := (B_j(s))_{j=0}^{681}, \quad i^2 = -1.$$

For $a + |\alpha| \leq 681$ odd,

$$\mathbb{E}[F_{\text{tr}}(W, X) \psi_a(W) \psi_\alpha(X)] = \frac{2(-1)^{(a+|\alpha|-1)/2}}{\pi \sqrt{a!}} \int_0^\infty e^{-s^2/2} s^{a-1} \prod_{j=1}^3 B_{\alpha_j}(s) ds. \quad (\text{C.12})$$

For even $a + |\alpha|$ the coefficient vanishes by $F_{\text{tr}}(-W, -X) = -F_{\text{tr}}(W, X)$ and [Eq. \(3.2\)](#). Reflection gives $\overline{B_j(s)} = i^j (-1)^j i^j B_j(s) = B_j(s)$, and Bessel's inequality gives $\|B(s)\|_2 \leq 1$. To prove [Eq. \(C.12\)](#), for $a \geq 0$ and $v \in \mathbb{R}$ use

$$\begin{aligned} q_a(v) &:= \mathbb{E}[\text{sign}(W + v) \psi_a(W)] = \frac{q_0^{(a)}(v)}{\sqrt{a!}}, \quad q_0(v) = 2 \int_0^v \phi(u) du, \\ q_a^{(l)}(v) &= \frac{2(-1)^{a+l-1}}{\sqrt{a!}} \phi(v) \text{He}_{a+l-1}(v) \quad (a, l \geq 0, \ a + l \geq 1), \\ q_a(v) &= \frac{2}{\pi \sqrt{a!}} \int_0^\infty e^{-s^2/2} s^{a-1} \Im(i^a e^{isv}) ds. \end{aligned} \quad (\text{C.13})$$

Conditioning on X and independence give

$$\begin{aligned} \mathbb{E}[F_{\text{tr}} \psi_a(W) \psi_\alpha(X)] &= \mathbb{E} \left[q_a \left(\sum_j r(X_j) \right) \psi_\alpha(X) \prod_j \mathbf{1}_{\{|X_j| \leq 73/10\}} \right], \\ \mathbb{E} \left[e^{i s \sum_j r(X_j)} \psi_\alpha(X) \prod_j \mathbf{1}_{\{|X_j| \leq 73/10\}} \right] &= i^{|\alpha|} \prod_j B_{\alpha_j}(s). \end{aligned}$$

Fubini follows from $\mathbb{E}|\psi_\alpha(X)| \leq 1$ and the integrability of $e^{-s^2/2}s^{a-1}$ for $a \geq 1$; for $a = 0$, use $|\sin(s \sum_j r(X_j))|/s \leq 3 \sup_{|x| \leq 73/10} |r(x)|$ when $\max_j |X_j| \leq 73/10$. The first two identities in Eq. (C.13) follow from

$$(\phi \text{He}_{a-1})' = -\phi \text{He}_a, \quad \frac{2}{\sqrt{a!}} \int_{-v}^{\infty} \text{He}_a(u) \phi(u) du = \frac{2(-1)^{a-1}}{\sqrt{a!}} \phi(v) \text{He}_{a-1}(v) \quad (a \geq 1).$$

For the Fourier identity, both sides at $a = 0$ vanish at $v = 0$, and

$$\frac{d}{dv} \left(\frac{2}{\pi} \int_0^\infty e^{-s^2/2} \frac{\sin(sv)}{s} ds \right) = \frac{2}{\pi} \int_0^\infty e^{-s^2/2} \cos(sv) ds = 2\phi(v).$$

The bounds $|\sin(sv)|/s \leq |v|$ and $|\partial_v^a(\sin(sv)/s)| \leq s^{a-1}$ for $a \geq 1$, multiplied by $e^{-s^2/2}$, are integrable in s , locally uniformly for real v .

2. Define the finite approximation $\hat{C}[a, b]$. Construct $\hat{C}$ by the operations below, with fixed summation orders and fixed choices of separate or combined multiplication and addition. Use the same resulting rational array in Eq. (C.11). Let s_ν, ω_ν , $1 \leq \nu \leq 768$, be the 24-point Gauss–Legendre nodes and weights on each $[k, k+1]$, $0 \leq k \leq 31$. Their weighted sum equals the integral on that interval for every polynomial of degree at most 47. Set

$$U_{a\nu} := \omega_\nu e^{-s_\nu^2/2} s_\nu^{a-1} / \sqrt{a!} \quad (0 \leq a \leq 681).$$

In interval arithmetic, each operation encloses the exact values by rounding the lower endpoint downward and the upper endpoint upward, at the stated precision. For $[a, b] \subset [0, 73/10]$, set $h := (b-a)/2$ and

$$M := \frac{(100/99)^{681/2} (10000/199)^{1/4}}{\sqrt{2\pi}} \exp \left(-\frac{50}{199} \max(0, a-h/4)^2 + 50(3h/4)^2 \right) \cdot \exp \left(24h \sum_{k=1}^{15} \frac{|r^{(k)}((a+b)/2)|}{(k-1)!} (5h/4)^{k-1} \right). \quad (\text{C.14})$$

Starting with $[0, 73/10]$, accept an interval only when its 512-bit interval evaluation proves $8(b-a)M2^{-64} < 10^{-10}(b-a)/(73/5)$; otherwise bisect it. Use the 32-point Gauss–Legendre rule, exact for polynomials of degree at most 63, on each interval. Index its positive nodes by j ; for the node x in $[a, b]$, define

$$v_j := 2\phi(x)(\psi_m(x))_{m=0}^{681} \int_a^b \prod_{\substack{y \neq x \\ y \text{ a quadrature node in } [a,b]}} \frac{t-y}{x-y} dt.$$

The integral is the quadrature weight. Reflection supplies the factor two: $i^{-m}(e^{isr(x)} + (-1)^m e^{-isr(x)}) = 2\Re(i^{-m} e^{isr(x)})$. At 512 bits, enclose the exact quadrature nodes and use these enclosures to compute v_j , $U_{a\nu}$, $4/\pi^2$, and $\Re(i^{-m} e^{is_\nu r(x)})$. Round the midpoint of each resulting enclosure to the nearest IEEE 754 binary64 number, which has at most 53 significant binary digits; at a tie, choose the number whose last stored binary digit is zero. Bound its conversion error against the whole enclosure. Hats denote computed values whose stored real and imaginary components are interpreted as exact rationals. Rounding an exact real result x to the nearest binary64 number has absolute error at most $2^{-53}|x| + 2^{-1074}$, provided that x is within the representable range. Compute $\hat{B}_{\nu m}$ by summing the products of $(\hat{v}_j)_m$ with the stored values of $\Re(i^{-m} e^{is_\nu r(x)})$, using at most one multiplication and one addition per term. Round each operation, or round a multiplication and its following addition together. Write $\hat{B}_\nu := (\hat{B}_{\nu m})_{m=0}^{681}$. Let $\hat{U}_{a\nu}$ denote the conversion before omitting entries below 10^{-18} ,

and, for each (ν, μ) , set $g(z) := \sum_{j=0}^{681} \hat{B}_{\nu j} \hat{B}_{\mu j} z^j$ with exact products. For odd $a + b \leq 681$, compute $\hat{C}[a, b]$ from

$$\frac{4}{\pi^2} \sum_{\mu: \hat{U}_{a\mu} \geq 10^{-18}} \hat{U}_{a\mu} \left(\sum_{\nu: \hat{U}_{a\nu} \geq 10^{-18}} \hat{U}_{a\nu} [z^b] \left(\sum_{j=0}^{681} \hat{B}_{\nu j} \hat{B}_{\mu j} z^j \right)^3 \right). \quad (\text{C.15})$$

Round the products defining g and append zeros to obtain 2048 entries. The exact formula approximated by the forward transform, two multiplications for each cube, and the inverse transform is

$$[z^b]g(z)^3 = \frac{1}{2048} \sum_{k=0}^{2047} g(e^{-2\pi i k/2048})^3 e^{2\pi i k b/2048} \quad (0 \leq b \leq 2043), \quad (\text{C.16})$$

since $\deg(g^3) \leq 2043$. First reverse the eleven binary digits of the indices $0, \dots, 2047$. For $k = 1, \dots, 11$, split each consecutive block of 2^k entries into halves and pair their j th entries, $0 \leq j < 2^{k-1}$, applying $(a, b) \mapsto (a + wb, a - wb)$ with $w = e^{-2\pi i j/2^k}$. Obtain $\hat{w}$ by rounding the real and imaginary parts of w from their 256-bit enclosure midpoints, with error $< 2^{-48}$ against each enclosure. The inverse conjugates these roots and ends with division by 2048. Take real parts, sum first over ν and then over μ in Eq. (C.15), and multiply by $\frac{4}{\pi^2}$ with the same rounding convention; replace negative results by zero and set $\hat{C}[a, b] = 0$ when $a + b$ is even.

3. Bound the error in $\hat{C}$. For $s > 0$ and real vectors v, w of equal dimension,

$$\sum_{a=0}^{681} e^{-s^2} s^{2a-2}/a! \leq s^{-2}, \quad \|v^{\otimes 3} - w^{\otimes 3}\|_2 \leq 3\|v - w\|_2 \max(\|v\|_2, \|w\|_2)^2.$$

Together with $\|B(s)\|_2 \leq 1$, the 512-bit bound $\sum_{\nu=1}^{768} \omega_\nu/s_\nu < 11.017652258306741$, and Claim 2, these inequalities give

$$\begin{aligned} & \left\| \left(\mathbb{E}[F_{\text{tr}} \psi_a(W) \psi_\alpha(X)] - \frac{2(-1)^{(a+|\alpha|-1)/2}}{\pi} \sum_{\nu=1}^{768} U_{a\nu} \prod_{j=1}^3 \hat{B}_{\nu\alpha_j} \right)_{\substack{a+|\alpha| \leq 681 \\ a+|\alpha| \text{ odd}}} \right\|_2 \\ & \leq 2.917598329143 \cdot 10^{-10} + 8.228746 \cdot 10^{-18} \\ & \quad + \frac{6}{\pi} \sum_{\nu=1}^{768} \frac{\omega_\nu}{s_\nu} (5.604655228362 \cdot 10^{-12}) (1 + 5.604655228362 \cdot 10^{-12})^2 \\ & < 4.096939258130 \cdot 10^{-10}. \end{aligned}$$

Since $\|F_{\text{tr}}\|_2 \leq 1$, Eq. (C.7) yields

$$\begin{aligned} & \sum_{\substack{a+b \leq 681 \\ a+b \text{ odd}}} \left| C_{\text{tr}}[a, b] - \frac{4}{\pi^2} \sum_{|\alpha|=b} \left(\sum_{\nu=1}^{768} U_{a\nu} \prod_{j=1}^3 \hat{B}_{\nu\alpha_j} \right)^2 \right| \\ & < 2(4.096939258130 \cdot 10^{-10}) + (4.096939258130 \cdot 10^{-10})^2 < 8.193878518 \cdot 10^{-10}. \\ & \sum_{|\alpha|=b} \left(\sum_{\nu=1}^{768} U_{a\nu} \prod_{j=1}^3 \hat{B}_{\nu\alpha_j} \right)^2 \\ & = \sum_{\nu, \mu=1}^{768} U_{a\nu} U_{a\mu} [z^b] \left(\sum_{j=0}^{681} \hat{B}_{\nu j} \hat{B}_{\mu j} z^j \right)^3. \end{aligned} \quad (\text{C.17})$$

Thus Claim 3 proves Eq. (C.10):

$$\sum_{a+b \leq 681} |\hat{C}[a, b] - C_{\text{tr}}[a, b]| \leq 8.193878518 \cdot 10^{-10} + 5.996940779766870 \cdot 10^{-8} = 6.078879564946870 \cdot 10^{-8}.$$

4. Evaluate the finite coefficient inequality. For $1 \leq m \leq 681$, Eq. (C.3) and Eq. (C.4) give

$$\begin{aligned} [t^m]H_{14}(t) &= \sum_{\substack{a, b \geq 0 \\ 3a+5b=m}} (-1)^a \frac{u_{ab}^2}{a!b!}, \\ [t^m]H_i(t) &= \frac{2}{\pi} \sum_{k=0}^{340} \frac{\binom{2k}{k}}{4^k(2k+1)} [t^m]P_i(t)^{2k+1} \quad (17 \leq i \leq 31), \end{aligned}$$

where u_{ab} is defined in Eq. (C.32) and $\mathbb{E}[F_*(X_1, X_2)\psi_a(X_1)\psi_b(X_2)] = u_{ab}/\sqrt{a!b!}$. To evaluate u_{ab} , define $J_j(x) := \int_x^\infty \text{He}_j(t)\phi(t) dt$ and use

$$\begin{aligned} \int_x^y \psi_j(t)\phi(t) dt &= \frac{J_j(x) - J_j(y)}{\sqrt{j!}} \quad (-\infty \leq x < y \leq \infty), \\ J_0(x) &= \frac{1}{2} \text{erfc}(x/\sqrt{2}), \quad J_j(x) = \phi(x) \text{He}_{j-1}(x) \quad (j \geq 1), \\ J_j(-\infty) &= \mathbf{1}_{\{j=0\}}, \quad J_j(+\infty) = 0. \end{aligned} \tag{C.18}$$

These identities are proved in Claim 4; evaluate them at 384 bits on the coordinate intervals of the rectangles where F_* is constant. For the remaining polynomial arithmetic, retain only degrees through 681: $P_i(0) = 0$ implies $[t^m]P_i^{2k+1} = 0$ for $m \leq 681$, $k > 340$, and products of nonnegative powers cannot lower degree. Interval evaluation at 256 bits, using the rational $\hat{C}[a, b]$, bounds the first term in Eq. (C.11) below by 0.561894649254995327808 and the subtracted sum above by $1.540129331402561534750 \cdot 10^{-5}$. Hence

$$0.561894649254995327808 - 1.540129331402561534750 \cdot 10^{-5} > 0.56187924796168130219.$$

□

Claim 2. *The quadratures in step 2 satisfy*

$$\begin{aligned} &\left\| \left(\mathbb{E}[F_{\text{tr}}\psi_a(W)\psi_\alpha(X)] - \frac{2(-1)^{(a+|\alpha|-1)/2}}{\pi} \sum_{\nu=1}^{768} U_{a\nu} \prod_{j=1}^3 B_{\alpha_j}(s_\nu) \right)_{\substack{a+|\alpha| \leq 681 \\ a+|\alpha| \text{ odd}}} \right\|_2 \\ &< 2.917598329143 \cdot 10^{-10} + 8.228746 \cdot 10^{-18}, \end{aligned} \tag{C.19}$$

and, for every $1 \leq \nu \leq 768$,

$$\|\hat{B}_\nu - B(s_\nu)\|_2 < 5.604655228362 \cdot 10^{-12}. \tag{C.20}$$

Proof. For an interval $[a, b]$ of length ℓ , consider the ellipse

$$z = \frac{a+b}{2} + \frac{b-a}{4}(2e^{i\theta} + 2^{-1}e^{-i\theta}), \quad 0 \leq \theta \leq 2\pi.$$

If the integrand is analytic inside and on this ellipse and has norm at most M there, the n -point Gauss–Legendre error is at most

$$\frac{\ell}{2} \frac{64M}{15(1-2^{-2})2^{2n}} = \frac{128}{45} \ell M 2^{-2n} \leq 8\ell M 2^{-2n}. \tag{C.21}$$

The scalar bound is [HT08, Theorem 2.1]. For complex Euclidean or Gaussian L_2 spaces, apply it after bounded complex-linear functionals v and use $\|x\| = \sup_{\|v\| \leq 1} |v(x)|$, with $\|v\| := \sup_{\|x\| \leq 1} |v(x)|$.

The outer quadrature. Apply Eq. (C.21) in Gaussian L_2 to

$$s \mapsto \frac{2 \sin(s(W + \sum_{j=1}^3 r(X_j)))}{\pi s} \prod_{j=1}^3 \mathbf{1}_{\{|X_j| \leq 73/10\}}.$$

Its singularity at zero is removable. Since r is bounded on $[-73/10, 73/10]$ and $\mathbb{E} e^{s|W|} < \infty$ for every real s , the displayed function is analytic in Gaussian L_2 . On the ellipses for $[k, k+1]$, $|\Im s| \leq 3/8$, so $|\sin(sv)/s| = |v \int_0^1 \cos(tsv) dt| \leq |v| e^{3|v|/8}$ for real v .

$$\begin{aligned} \mathbb{E}[e^{3|r(X_1)|/4} \mathbf{1}_{\{|X_1| \leq 73/10\}}] &< 2.262302379287, \\ \mathbb{E}[r(X_1)^2 e^{3|r(X_1)|/4} \mathbf{1}_{\{|X_1| \leq 73/10\}}] &< 2023.738286188, \\ \mathbb{E} e^{3|W|/4} &\leq 2e^{9/32}, \quad \mathbb{E} W^2 e^{3|W|/4} \leq (25/8)e^{9/32}. \end{aligned}$$

By Bessel's inequality, retaining the Hermite coefficients with $a + |\alpha| \leq 681$ and $a + |\alpha|$ odd cannot increase the L_2 norm. Independence and Minkowski's inequality therefore bound the contribution of the quadrature on $[0, 32]$ to Eq. (C.19) by

$$\begin{aligned} &8 \cdot 32 \cdot 2^{-48} \frac{2\sqrt{2}}{\pi} e^{9/64} \left(\frac{5}{4} (2.262302379287)^{3/2} + 3\sqrt{2023.738286188} (2.262302379287) \right) \\ &< 2.917598329143 \cdot 10^{-10}. \end{aligned}$$

For the first two moments, partition $[0, 73/10]$ into intervals of length 10^{-3} . On each $[a, b]$, let R be the upper endpoint of the modulus of the 512-bit interval polynomial evaluation $r([a, b])$ and sum

$$\begin{aligned} 2 \int_a^b e^{3|r(x)|/4} \phi(x) dx &\leq 2(b-a) \phi(a) e^{3R/4}, \\ 2 \int_a^b r(x)^2 e^{3|r(x)|/4} \phi(x) dx &\leq 2(b-a) \phi(a) R^2 e^{3R/4}. \end{aligned}$$

The factor two uses oddness of r and evenness of ϕ . For $s \geq 32$, Bessel's inequality and a geometric sum give

$$\sum_{|\alpha| \leq 681-a} \left| \prod_j B_{\alpha_j}(s) \right|^2 \leq 1, \quad \sum_{a=0}^{681} \frac{s^{2a}}{a!} \leq \frac{s^{1362}}{681!} \sum_{j=0}^{681} (681/s^2)^j \leq \frac{1024s^{1362}}{343 \cdot 681!}.$$

Minkowski's inequality bounds the integral over $s > 32$ by

$$\begin{aligned} &\left\| \left(\frac{2}{\pi\sqrt{a!}} \int_{32}^{\infty} e^{-s^2/2} s^{a-1} \prod_{j=1}^3 B_{\alpha_j}(s) ds \right)_{\substack{a+|\alpha| \leq 681 \\ a+|\alpha| \text{ odd}}} \right\|_2 \\ &\leq \frac{64}{\pi\sqrt{343 \cdot 681!}} \int_{32}^{\infty} e^{-s^2/2} s^{680} ds \leq \frac{64e^{-512} 32^{681}}{345\pi\sqrt{343 \cdot 681!}} < 3 \cdot 10^{-18} < 8.228746 \cdot 10^{-18}. \end{aligned} \tag{C.22}$$

The second inequality follows by integrating

$$-\frac{d}{ds}(e^{-s^2/2} s^{679}) = e^{-s^2/2} s^{680} (1 - 679/s^2) \geq \frac{345}{1024} e^{-s^2/2} s^{680} \quad (s \geq 32).$$

The computation of $\hat{B}$. On the ellipse over $[a, b]$, $|z - (a + b)/2| \leq 5h/4$, $|\Im z| \leq 3h/4$, and $|\Re z| \geq \max(0, a - h/4)$. Mehler's identity and the Taylor expansion of r give

$$\begin{aligned} \sum_{j=0}^{681} |\psi_j(z)|^2 &\leq (100/99)^{681} \sum_{j \geq 0} (99/100)^j |\psi_j(z)|^2 \\ &= (100/99)^{681} (10000/199)^{1/2} e^{(99/199)(\Re z)^2 + 99(\Im z)^2}, \\ |\Im r(z)| &\leq |\Im z| \sum_{k=1}^{15} \frac{|r^{(k)}((a+b)/2)|}{(k-1)!} |z - (a+b)/2|^{k-1}. \end{aligned}$$

Multiplying the square root of the first bound by $|\phi(z)e^{isr(z)}| \leq (2\pi)^{-1/2} e^{-(\Re z)^2/2 + (\Im z)^2/2 + 32|\Im r(z)|}$ yields $\|(\phi(z)\psi_j(z)e^{isr(z)})_{j=0}^{681}\|_2 \leq M$ for $0 \leq s \leq 32$, with M as in Eq. (C.14). The bisection accepts 50 intervals, giving 1600 positive nodes and

$$2 \sum_{[a,b]} 8(b-a)M2^{-64} < 3.288380170604 \cdot 10^{-12}. \quad (\text{C.23})$$

The sum is over the accepted intervals. By reflection and Eq. (C.21), it bounds

$$\left\| B(s) - \left(\sum_x (v_j)_m \Re(i^{-m} e^{isr(x)}) \right)_{m=0}^{681} \right\|_2 \quad (0 \leq s \leq 32),$$

where j indexes x as in step 2. Put $\gamma_k := k2^{-53}/(1 - k2^{-53})$ for integers $0 \leq k < 2^{53}$. For an n -term dot product, $2n < 2^{53}$, using at most n multiplications and n additions with the stated rounding and no overflow, the error is at most γ_{2n} times the sum of the absolute values of its summands, apart from the additive 2^{-1074} rounding errors. Rounding a multiplication and addition together does not increase this bound. For every quadrature node and $0 \leq m \leq 681$, the input enclosures verify

$$\left| \Re(\widehat{i^{-m} e^{is_\nu r(x)}}) - \Re(i^{-m} e^{is_\nu r(x)}) \right| < 2^{-48}.$$

The same 512-bit evaluation gives

$$\sum_j \|v_j - \hat{v}_j\|_2 < 3.050177189723512 \cdot 10^{-16}, \quad \sum_j \|v_j\|_2 < 6.454332802795005.$$

$$\begin{aligned} \|\hat{B}_\nu - B(s_\nu)\|_2 &\leq 3.288380170604 \cdot 10^{-12} + \sum_j \|v_j - \hat{v}_j\|_2 \\ &\quad + (2^{-48} + \gamma_{3200}(1 + 2^{-48})) \sum_j (\|v_j\|_2 + \|v_j - \hat{v}_j\|_2) + 10^{-290} \\ &< 5.604655228362 \cdot 10^{-12}. \end{aligned} \quad (\text{C.24})$$

□

Claim 3.

$$\sum_{\substack{a+b \leq 681 \\ a+b \text{ odd}}} \left| \hat{C}[a, b] - \frac{4}{\pi^2} \sum_{\nu, \mu=1}^{768} U_{a\nu} U_{a\mu} [z^b] \left(\sum_{j=0}^{681} \hat{B}_{\nu j} \hat{B}_{\mu j} z^j \right)^3 \right| \leq 5.996940779766870 \cdot 10^{-8}. \quad (\text{C.25})$$

Proof. For each (ν, μ) , Cauchy–Schwarz and Eq. (C.20) give

$$\begin{aligned} \sum_j |[z^j]g(z)| &\leq \|\hat{B}_{\nu}\|_2 \|\hat{B}_{\mu}\|_2 \leq (1 + 5.604656 \cdot 10^{-12})^2, \\ \sum_j |[z^j]g(z)|^3 &\leq \left(\sum_j |[z^j]g(z)| \right)^3. \end{aligned}$$

Here and below the weight indices range over $0 \leq a \leq 681$, $1 \leq \nu, \mu \leq 768$. The input enclosures and stored numbers give, at 256 bits,

$$\begin{aligned} \sum_{a,\nu} |U_{a\nu} - \hat{U}_{a\nu}| &< 1.092870488954584 \cdot 10^{-15}, \quad \sum_{a,\nu} |\hat{U}_{a\nu}| < 26.904610216896492, \\ \sum_a \left(\sum_{\nu} |\hat{U}_{a\nu}| \right)^2 &< 61.515376731920361, \quad \left| \frac{4}{\pi^2} - \widehat{\frac{4}{\pi^2}} \right| < 1.586515951775777 \cdot 10^{-17}, \quad (\text{C.26}) \\ \sum_{a,\nu: \hat{U}_{a\nu} < 10^{-18}} |\hat{U}_{a\nu}| &< 2.655682807169783 \cdot 10^{-15}, \quad \widehat{\frac{4}{\pi^2}} = \frac{228155022448185}{562949953421312}. \end{aligned}$$

For conversion of the weights and $4/\pi^2$,

$$\begin{aligned} \sum_{a=0}^{681} \sum_{\nu,\mu=1}^{768} \left| \frac{4}{\pi^2} U_{a\nu} U_{a\mu} - \widehat{\frac{4}{\pi^2}} \hat{U}_{a\nu} \hat{U}_{a\mu} \right| &\leq \frac{4}{\pi^2} \left(\sum_{a,\nu} |U_{a\nu} - \hat{U}_{a\nu}| \right) \left(2 \sum_{a,\nu} |\hat{U}_{a\nu}| + \sum_{a,\nu} |U_{a\nu} - \hat{U}_{a\nu}| \right) \\ &\quad + \left| \frac{4}{\pi^2} - \widehat{\frac{4}{\pi^2}} \right| \sum_a \left(\sum_{\nu} |\hat{U}_{a\nu}| \right)^2 \\ &< 8.386939012631 \cdot 10^{-14}. \end{aligned} \quad (\text{C.27})$$

Multiplication by $(1 + 5.604656 \cdot 10^{-12})^6$ bounds the resulting coefficient error. Omitting entries below 10^{-18} contributes at most

$$\begin{aligned} &\widehat{\frac{4}{\pi^2}} (1 + 5.604656 \cdot 10^{-12})^6 \sum_a \sum_{\substack{\nu,\mu \\ \hat{U}_{a\nu} < 10^{-18} \text{ or } \hat{U}_{a\mu} < 10^{-18}}} |\hat{U}_{a\nu} \hat{U}_{a\mu}| \\ &\leq \widehat{\frac{4}{\pi^2}} (1 + 5.604656 \cdot 10^{-12})^6 \sum_a \left[2 \left(\sum_{\nu} |\hat{U}_{a\nu}| \right) \left(\sum_{\nu: \hat{U}_{a\nu} < 10^{-18}} |\hat{U}_{a\nu}| \right) + \left(\sum_{\nu: \hat{U}_{a\nu} < 10^{-18}} |\hat{U}_{a\nu}| \right)^2 \right] \\ &< 5.791527838 \cdot 10^{-14}. \end{aligned}$$

In each transform in Eq. (C.16), the eleven stages $(a, b) \mapsto (a + wb, a - wb)$ satisfy

$$|a + wb|^2 + |a - wb|^2 = 2(|a|^2 + |b|^2) \quad (|w| = 1).$$

For $a, b \in \mathbb{C}$, let $\widehat{ab}$ be their product computed with four real multiplications and two real additions. Apart from the additive 2^{-1074} errors treated below,

$$|\widehat{ab} - ab| \leq \sqrt{2}(2 \cdot 2^{-53} + 2^{-106})|a||b| \leq 3 \cdot 2^{-53}|a||b|.$$

The component errors in w therefore give

$$|\widehat{wb} - wb| \leq (\sqrt{2}2^{-48} + 3 \cdot 2^{-53}(1 + \sqrt{2}2^{-48}))|b| \leq 50 \cdot 2^{-53}|b|.$$

The two final additions give stage error at most $(51 + 50 \cdot 2^{-53})2^{-53}\sqrt{2(|a|^2 + |b|^2)}$, where $51 + 50 \cdot 2^{-53} < 100$. For $0 \leq k \leq 11$, $(1 + 100 \cdot 2^{-53})^k - 1 \leq \gamma_{100k}$. Including rounded input products, the Euclidean error after k stages is at most

$$2^{k/2}(\gamma_{100k}(1 + 2^{-53}) + 2^{-53}) \sum_j |[z^j]g(z)| \leq 5.53175 \cdot 10^{-12} \sum_j |[z^j]g(z)|.$$

In exact arithmetic, each intermediate entry is a sum of distinct input coefficients with multipliers of modulus one, so its modulus is at most $\sum_j |[z^j]g(z)|$. For coordinatewise cubes of complex vectors,

$$\|v^3 - w^3\|_2 \leq 3 \max(\|v\|_\infty, \|w\|_\infty)^2 \|v - w\|_2.$$

For $(zz)z$, the product bound gives rounding error at most $((1 + 3 \cdot 2^{-53})^2 - 1)|z|^3 \leq \gamma_{20}|z|^3$. For the computed forward-transform vector $v \in \mathbb{C}^{2048}$, rounding these cubes contributes Euclidean error at most $\gamma_{20}\sqrt{2048}\|v\|_\infty^3$. The Euclidean operator norm of the inverse transform is $1/\sqrt{2048}$, and $\|v\|_1 \leq \sqrt{2048}\|v\|_2$. The sum of absolute coefficient errors is therefore at most

$$\begin{aligned} & \left(\sum_j |[z^j]g(z)| \right)^3 (3(1 + 5.53175 \cdot 10^{-12})^2 (5.53175 \cdot 10^{-12}) \\ & \quad + \sqrt{2048}(\gamma_{20} + \gamma_{1100}(1 + \gamma_{20}))(1 + 5.53175 \cdot 10^{-12})^3) < 2.222247 \cdot 10^{-11} \end{aligned} \quad (\text{C.28})$$

and, including the additive rounding errors, at most

$$10000 \cdot 2048 \cdot 2^{-53} \left(\sum_j |[z^j]g(z)| \right)^3 + 10^{-290} < 2.273736754509 \cdot 10^{-9}. \quad (\text{C.29})$$

For each (ν, μ) , the input products, two transforms, cubes, and inverse normalization use at most

$$682 + 22 \cdot 1024 \cdot 10 + 2048(12 + 2) < 2^{22}$$

real operations. The bounds for $0 \leq k \leq 11$ give forward values of modulus at most

$$(1 + 5.53175 \cdot 10^{-12})(1 + 5.604656 \cdot 10^{-12})^2 + 4 \cdot 2^{22} \cdot 2^{11} \cdot 2^{-1074} < \frac{3}{2}.$$

An error in a real product passes through at most two remaining real additions and is copied to the two outputs, with Euclidean amplification $\sqrt{2}(1 + 2^{-53})^2 < 4$. With local rounding errors separated, each remaining linear stage map has operator norm < 2 . The forward and cube bounds give inverse inputs of modulus < 8 , and all inverse intermediate magnitudes are bounded by

$$8 \cdot 2048(1 + 100 \cdot 2^{-53})^{11} + 4 \cdot 2^{22} \cdot 2^{11} \cdot 2^{-1074} < 2^{15} < 2^{16}.$$

Thus the transform and cube operations do not overflow. With the relative rounding factors fixed, each computed product is bilinear over $\mathbb{R}$, and the product bound gives input-error amplification at most

$$3(1 + 3 \cdot 2^{-53})^2 \left(\frac{3}{2} \right)^2 < 12 \quad (|z| < 3/2)$$

for the two products forming z^3 . Errors within these multiplications have amplification at most $4 \cdot 12$. Across at most 22 transform stages and the final division by 2048, a local error contributes at most

$$4 \cdot 2^{22} \cdot 12 \cdot \frac{\sqrt{2048}}{2048} < 2^{24} < 2^{40}$$

times its magnitude to the sum of absolute coefficient errors. Errors in the final division have amplification one. The total additive error is $< 2^{22}2^{-1074}2^{40} < 10^{-300}$. All B dot products and weighted sums together use fewer than $2 \cdot 1600 \cdot 768 \cdot 682 + 682^2(2 \cdot 768^2 + 8 \cdot 768) < 2^{40}$ operations. Their intermediate magnitudes are bounded, respectively, by $2 \sum_j (\|v_j\|_2 + \|v_j - \hat{v}_j\|_2) < 16$ and $2(1 + \sum_{a,\nu} |\hat{U}_{a\nu}|)^2 < 2^{12}$, so they do not overflow. A local error reaches its dot-product or weighted-sum output through at most two weight multiplications, the final factor, and 1185792 rounded operations. Its amplification is at most

$$(1 + 2^{-53})^{1185792} \max\left(1, \frac{4}{\pi^2}\right) \left(1 + \sum_{a,\nu} |\hat{U}_{a\nu}|\right)^2 < 2^{12}.$$

Their additive errors therefore sum to less than $2^{40}2^{12}2^{-1074} < 10^{-300} < 10^{-290}$. Taking real parts cannot increase error, since $|\Re z| \leq |z|$. Using Eq. (C.27) with the larger coefficient bound from Eq. (C.29), the conversion error is at most

$$(8.386939012631 \cdot 10^{-14})((1 + 5.604656 \cdot 10^{-12})^6 + 2.273736754509 \cdot 10^{-9}) < 8.386939032 \cdot 10^{-14}.$$

After multiplication by the weights, the error in the computed cube coefficients and the error in the final sums are respectively bounded by

$$\begin{aligned} & \frac{4}{\pi^2} \sum_a \left(\sum_\nu |\hat{U}_{a\nu}| \right)^2 (2.273736754509 \cdot 10^{-9}) < 5.668708385 \cdot 10^{-8}, \\ & \gamma_{1185792} \frac{4}{\pi^2} \sum_a \left(\sum_\nu |\hat{U}_{a\nu}| \right)^2 ((1 + 5.604656 \cdot 10^{-12})^6 + 2.273736754509 \cdot 10^{-9}) + 10^{-290} \\ & < 3.282182163 \cdot 10^{-9}, \end{aligned}$$

The inner sums, outer sum, and final multiplication use at most $768(2 \cdot 768) + 2 \cdot 768 + 1 \leq 2 \cdot 768^2 + 8 \cdot 768 = 1185792$ real operations per coefficient. Adding the conversion, omission, transform, and summation bounds gives

$$\begin{aligned} & 8.386939032 \cdot 10^{-14} + 5.791527838 \cdot 10^{-14} + 5.668708385 \cdot 10^{-8} + 3.282182163 \cdot 10^{-9} \\ & = 5.996940779766870 \cdot 10^{-8}. \end{aligned}$$

The exact entries are nonnegative by Eq. (C.17); $|\max(x, 0) - y| \leq |x - y|$ for $y \geq 0$ shows that replacing negative entries by zero cannot increase error. $\square$

C.2 Bounds above degree 681 for $H_{14}, H_{17}, \dots, H_{31}$

Lemma C.2.

$$\lambda_{14} \sum_{m>681} |[t^m]H_{14}(t)| + \sum_{i=17}^{31} \lambda_i \sum_{m>681} |[t^m]H_i(t)| < 0.000003878150168295. \quad (\text{C.30})$$

Proof. Claims 4 and 5 give

$$\begin{aligned} & \lambda_{14} \sum_{m>681} |[t^m]H_{14}(t)| + \sum_{i=17}^{31} \lambda_i \sum_{m>681} |[t^m]H_i(t)| \\ & < (1.254728050780 + 2.623422117515) \cdot 10^{-6} = 0.000003878150168295. \end{aligned}$$

$\square$

Claim 4.

$$\lambda_{14} \sum_{m>681} |[t^m]H_{14}(t)| < 1.254728050780 \cdot 10^{-6} < 1.254729 \cdot 10^{-6}. \quad (\text{C.31})$$

Proof. The formulas for J_j in Eq. (C.18) follow from

$$\frac{d}{dx} \operatorname{erfc}(x/\sqrt{2}) = -2\phi(x), \quad (\phi \operatorname{He}_{j-1})' = -\phi \operatorname{He}_j, \quad \lim_{x \rightarrow \pm\infty} \phi(x) \operatorname{He}_{j-1}(x) = 0 \quad (j \geq 1).$$

Define $u_{ij} := \mathbb{E}[F_*(X_1, X_2) \operatorname{He}_i(X_1) \operatorname{He}_j(X_2)]$. By Eq. (3.2), $u_{ij} = 0$ for $i + j$ even. On the strip $k/256 \leq y < (k+1)/256$, the definition of F_* gives

$$\begin{aligned} \int_{\mathbb{R}} F_*(x, y) \operatorname{He}_i(x) \phi(x) dx &= \mathbf{1}_{\{i=0\}} + 2 \sum_a (-1)^a J_i(r_{k,a}), \\ \int_{\mathbb{R}} \operatorname{sign}(x) \operatorname{He}_i(x) \phi(x) dx &= 2 \mathbf{1}_{\{i \text{ odd}\}} J_i(0). \end{aligned}$$

For $i + j$ odd, symmetry and $F_*(x, y) = -\operatorname{sign}(x)$ for $y \geq 8$ yield

$$\begin{aligned} u_{ij} &= 2 \int_0^\infty \int_{\mathbb{R}} F_*(x, y) \operatorname{He}_i(x) \operatorname{He}_j(y) \phi(x) \phi(y) dx dy \\ &= 2 \sum_{k=0}^{2047} \left(\mathbf{1}_{\{i=0\}} + 2 \sum_a (-1)^a J_i(r_{k,a}) \right) [J_j(k/256) - J_j((k+1)/256)] \\ &\quad - 4 \mathbf{1}_{\{i \text{ odd}\}} J_i(0) J_j(8). \end{aligned} \quad (\text{C.32})$$

By Eq. (C.3) and Parseval,

$$\sum_{m>681} |[t^m]H_{14}(t)| \leq \sum_{3i+5j>681} \frac{u_{ij}^2}{i!j!} = 1 - \sum_{3i+5j \leq 681} \frac{u_{ij}^2}{i!j!}. \quad (\text{C.33})$$

Evaluating Eq. (C.32) with 384-bit interval arithmetic gives

$$\begin{aligned} \#\{(i, j) \in \mathbb{Z}_{\geq 0}^2 : i + j \text{ odd}, 3i + 5j \leq 681\} &= 7843, \\ \sum_{3i+5j \leq 681} \frac{u_{ij}^2}{i!j!} &> 0.9648417014661795, \quad \lambda_{14}(1 - 0.9648417014661795) < 1.254728050780 \cdot 10^{-6}. \end{aligned}$$

□

Claim 5.

$$\sum_{i=17}^{31} \lambda_i \sum_{m>681} |[t^m]H_i(t)| < 2.623422117515 \cdot 10^{-6} < 2.623423 \cdot 10^{-6}. \quad (\text{C.34})$$

Proof. For $17 \leq i \leq 31$, Eq. (C.4), Eq. (C.8), and

$$\arcsin z = \sum_{k \geq 0} \frac{\binom{2k}{k}}{4^k(2k+1)} z^{2k+1} \quad (|z| \leq 1) \quad (\text{C.35})$$

give

$$|[t^m]H_i(t)| \leq [t^m] \frac{2}{\pi} \arcsin P_{i,+}(t), \quad \sum_{m \geq 0} [t^m] \frac{2}{\pi} \arcsin P_{i,+}(t) = \frac{2}{\pi} \arcsin P_{i,+}(1) \leq 1.$$

Consequently,

$$\sum_{m>681} |[t^m]H_i(t)| \leq 1 - \sum_{m=1}^{681} [t^m] \frac{2}{\pi} \arcsin P_{i,+}(t). \quad (\text{C.36})$$

For $19 \leq i \leq 31$, evaluate at 256 bits the finite sum

$$\begin{aligned} \sum_{m=1}^{681} [t^m] \frac{2}{\pi} \arcsin P_{i,+}(t) &= \frac{2}{\pi} \sum_{k=0}^{340} \frac{\binom{2k}{k}}{4^k(2k+1)} \sum_{m=1}^{681} [t^m] P_{i,+}(t)^{2k+1}, \\ [t^m] P_{i,+}^{2k+1} &= 0 \quad (m < 2k+1). \end{aligned}$$

For $i = 17, 18$, set

$$\alpha := [t]P_i > 0, \quad \beta := -[t^3]P_i > 0, \quad (R, \eta) := \begin{cases} (6/5, 19/20), & i = 17, \\ (21/20, 49/50), & i = 18. \end{cases}$$

For $z = Re^{i\theta}$ and $x = \cos \theta$,

$$\Re P_i(z) = (\alpha R + 3\beta R^3)x - 4\beta R^3 x^3.$$

Its squared endpoint and interior critical values satisfy

$$(\alpha R - \beta R^3)^2 < \eta^2, \quad \frac{(\alpha R + 3\beta R^3)^3}{27\beta R^3} < \eta^2 \quad \text{if} \quad \frac{\alpha R + 3\beta R^3}{12\beta R^3} < 1.$$

Thus $\sup_{|z| \leq R} |\Re P_i(z)| < \eta$ by the maximum principle. On $|\Re w| < 1$, take the analytic branch of $\arcsin$ with $\arcsin(0) = 0$, $\arcsin'(0) = 1$. For $w = x + iy$, $|x| \leq \eta$,

$$\begin{aligned} |1 - w^2|^2 &= (1 - x^2)^2 + 2(1 + x^2)y^2 + y^4 \geq (1 - \eta^2)^2, \\ |\arcsin w| &= \left| \int_0^1 \frac{w}{\sqrt{1 - t^2 w^2}} dt \right| \leq \frac{|w|}{\sqrt{1 - \eta^2}}. \end{aligned}$$

Cauchy's estimate and oddness give

$$|[t^m]H_i(t)| \leq \frac{2(\alpha R + \beta R^3)}{\pi \sqrt{1 - \eta^2}} R^{-m}, \quad [t^{2m}]H_i(t) = 0,$$

and hence

$$\sum_{m>681} |[t^m]H_i(t)| \leq \frac{2(\alpha R + \beta R^3)}{\pi \sqrt{1 - \eta^2}} \frac{R^{-683}}{1 - R^{-2}} < \begin{cases} 6.982 \cdot 10^{-54}, & i = 17, \\ 1.223 \cdot 10^{-13}, & i = 18. \end{cases} \quad (\text{C.37})$$

These calculations use the normalized coefficients [Eq. \(4.2\)](#). Summing the 256-bit interval bounds [Eq. \(C.36\)](#) and [Eq. \(C.37\)](#) with weights λ_i gives [Eq. \(C.34\)](#). $\square$

C.3 Bounds above degree 681 for $H_1, \dots, H_{13}$

Lemma C.3.

$$\sum_{i=1}^{13} \lambda_i \sum_{m>681} |[t^m]H_i(t)| < 0.000034097111321496. \quad (\text{C.38})$$

Proof. For $1 \leq i \leq 13$, Parseval and [Eq. \(C.8\)](#) give

$$\sum_{m \geq 0} [t^m] \sum_{a, b \geq 0} C_{\text{tr}}[a, b] P_{i,+}(t)^a t^b = \sum_{a, b \geq 0} C_{\text{tr}}[a, b] P_{i,+}(1)^a \leq \mathbb{E}[F_{\text{tr}}(W, X)^2] \leq 1.$$

Therefore Eq. (C.2), Eq. (C.8), Eq. (C.6), and Eq. (C.10) imply

$$\begin{aligned}
\sum_{m>681} |[t^m]H_i(t)| &\leq \sum_{m>681} [t^m] \sum_{a,b \geq 0} C_{\text{tr}}[a,b] P_{i,+}(t)^a t^b + \sum_{a,b \geq 0} |C[a,b] - C_{\text{tr}}[a,b]| \\
&\leq 1 - \sum_{m=0}^{681} [t^m] \sum_{a+b \leq 681} \hat{C}[a,b] P_{i,+}(t)^a t^b + \sum_{a+b \leq 681} |\hat{C}[a,b] - C_{\text{tr}}[a,b]| \\
&\quad + \sum_{a,b \geq 0} |C[a,b] - C_{\text{tr}}[a,b]| \\
&\leq 1 - \sum_{m=0}^{681} [t^m] \sum_{a+b \leq 681} \hat{C}[a,b] P_{i,+}(t)^a t^b \\
&\quad + 6.078879564946870 \cdot 10^{-8} + 2\sqrt{3 \operatorname{erfc}\left(\frac{73}{10\sqrt{2}}\right)}.
\end{aligned} \tag{C.39}$$

Evaluating the finite sum at 256 bits using the exact rational $\hat{C}[a,b]$ from Claim 1 gives the following upper bounds for $\lambda_i \sum_{m>681} |[t^m]H_i(t)|$:

i	upper bound	i	upper bound
1	$2.364541 \cdot 10^{-7}$	8	$6.366458 \cdot 10^{-6}$
2	$1.243556 \cdot 10^{-7}$	9	$3.079728 \cdot 10^{-7}$
3	$3.261885 \cdot 10^{-7}$	10	$3.796353 \cdot 10^{-6}$
4	$2.277905 \cdot 10^{-6}$	11	$2.153852 \cdot 10^{-7}$
5	$3.923603 \cdot 10^{-6}$	12	$1.933337 \cdot 10^{-6}$
6	$1.454752 \cdot 10^{-6}$	13	$5.492683 \cdot 10^{-7}$
7	$1.258509 \cdot 10^{-5}$		

Summing the interval bounds before rounding to the displayed decimals gives Eq. (C.38). $\square$

C.4 Bounds above degree 681 for $\sum_{i \in \{0,15,16\}} \lambda_i H_i$

Lemma C.4.

$$\sum_{m>681} \left| [t^m] \sum_{i \in \{0,15,16\}} \lambda_i H_i(t) \right| < 0.000007875733380418. \tag{C.40}$$

Proof. Define

$$\begin{aligned}
\chi(u) &:= 1 - 462u^6 + 1980u^7 - 3465u^8 + 3080u^9 - 1386u^{10} + 252u^{11}, \\
\tilde{r}(x) &:= \begin{cases} r(x), & |x| \leq 27/4, \\ \operatorname{sign}(x) \left(r(27/4) + \int_0^{\min\{|x|-27/4, 1/4\}} r'(27/4+u) \chi(4u) du \right), & |x| > 27/4. \end{cases}
\end{aligned} \tag{C.41}$$

For $i \in \{0, 15, 16\}$ and real $t \in [-1, 1]$, let

$$\tilde{F}(w, x) := \operatorname{sign} \left(w + \sum_{j=1}^3 \tilde{r}(x_j) \right), \quad \tilde{H}_i(t) := \mathbb{E}[\tilde{F}(W, X) \tilde{F}(W', X')],$$

where $(W, W'), (X_j, X'_j)$ are independent centered jointly Gaussian pairs with unit marginal variances and $\mathbb{E}[WW'] = P_i(t)$, $\mathbb{E}[X_j X'_j] = -t$, $1 \leq j \leq 3$. Set $\mathcal{D} := t d/dt$ and $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$, parametrized by $t = e^{i\theta}$ with measure $d\theta/(2\pi)$.

Claim 6. The function $\tilde{r}$ is bounded and C^6 , with $\tilde{r}^{(j)}(x) = 0$ for $|x| > 7$, $1 \leq j \leq 6$, and

$$\sum_{m \geq 0} \left| [t^m] \sum_{i \in \{0,15,16\}} \lambda_i (H_i - \tilde{H}_i) \right| \leq 4\sqrt{\Pr(F \neq \tilde{F})} \sum_{i \in \{0,15,16\}} \lambda_i < 8.769183 \cdot 10^{-7}. \quad (\text{C.42})$$

Claim 7. The Taylor series of each $\tilde{H}_i$ converges absolutely and uniformly on $|t| \leq 1$, its restriction to $\mathbb{T}$ is C^4 , and $d^4 \tilde{H}_i(e^{i\theta})/d\theta^4 = \mathcal{D}^4 \tilde{H}_i(e^{i\theta})$. Moreover,

$$\frac{\left\| \mathcal{D}^4 \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right\|_{L_2(\mathbb{T})}}{\sqrt{14 \cdot 681^7}} + 4\sqrt{\Pr(F \neq \tilde{F})} \sum_{i \in \{0,15,16\}} \lambda_i < 0.000007875733380418. \quad (\text{C.43})$$

By Claim 7, four integrations by parts and Parseval give

$$\sum_{m \geq 0} m^8 \left| [t^m] \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right|^2 = \left\| \mathcal{D}^4 \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right\|_{L_2(\mathbb{T})}^2.$$

Oddness of $\tilde{r}$ and Eq. (3.2) imply that the even coefficients vanish. Thus Cauchy–Schwarz and

$$\sum_{\substack{m > 681 \\ m \text{ odd}}} m^{-8} \leq \frac{1}{2} \int_{681}^{\infty} x^{-8} dx = (14 \cdot 681^7)^{-1}$$

give

$$\sum_{m > 681} \left| [t^m] \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right| \leq \frac{\left\| \mathcal{D}^4 \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right\|_{L_2(\mathbb{T})}}{\sqrt{14 \cdot 681^7}}.$$

Adding Eq. (C.42) and using Eq. (C.43) proves the lemma. $\square$

Proof of Claim 6. We use the inequalities

$$r(27/4) > 7, \quad r^{(j)}(27/4) > 0 \quad (1 \leq j \leq 15), \quad |r(x)| < 33/20 \quad (|x| \leq 7/2). \quad (\text{C.44})$$

For $s \geq 0$, Eq. (C.41) and $0 \leq \chi(u) \leq 1$ for $0 \leq u \leq 1$ give

$$r'(27/4 + s) = \sum_{j=1}^{15} \frac{r^{(j)}(27/4)}{(j-1)!} s^{j-1} > 0, \\ \min\{r(27/4 + s), \tilde{r}(27/4 + s)\} \geq r(27/4) > 7.$$

Therefore

$$\max_j |X_j| \leq 27/4 \implies F = \tilde{F}, \\ |X_i| > 27/4, \quad \max_{j \neq i} |X_j| \leq 7/2, \quad |W| \leq 37/10 \implies F = \tilde{F} = \text{sign}(X_i),$$

because $7 = 2(33/20) + 37/10$. Hence

$$\{F \neq \tilde{F}\} \subseteq \bigcup_{i \neq j} \{|X_i| > 27/4, |X_j| > 7/2\} \cup \bigcup_i \{|X_i| > 27/4, |W| > 37/10\}, \quad 1 \leq i, j \leq 3.$$

Independence and the union bound give

$$\Pr(F \neq \tilde{F}) \leq 6 \operatorname{erfc}\left(\frac{27}{4\sqrt{2}}\right) \operatorname{erfc}\left(\frac{7}{2\sqrt{2}}\right) + 3 \operatorname{erfc}\left(\frac{27}{4\sqrt{2}}\right) \operatorname{erfc}\left(\frac{37}{10\sqrt{2}}\right). \quad (\text{C.45})$$

Parseval and Eq. (C.7) yield

$$\begin{aligned} & \sum_{a \geq 0, \beta \in \{0,1,\dots\}^3} \left| (\mathbb{E}[F\psi_a(W)\psi_\beta(X)])^2 - (\mathbb{E}[\tilde{F}\psi_a(W)\psi_\beta(X)])^2 \right| \\ & \leq \|F - \tilde{F}\|_2 \|F + \tilde{F}\|_2 \leq 4\sqrt{\Pr(F \neq \tilde{F})}. \end{aligned}$$

Together with Eq. (C.8) and Eq. (3.3), this proves Eq. (C.42); the numerical evaluation of Eq. (C.45) uses interval rounding at 256 bits.

The conclusions

$$0 \leq \chi(u) \leq 1 \quad (0 \leq u \leq 1), \quad \tilde{r} \in C^6(\mathbb{R})$$

follow from

$$\chi(0) = 1, \quad \chi(1) = 0, \quad \chi'(u) = -2772u^5(1-u)^5, \quad \chi^{(j)}(0) = \chi^{(j)}(1) = 0 \quad (1 \leq j \leq 5).$$

For $c = -7/2 + (2k+1)/32$, $0 \leq k < 112$, the Taylor bounds

$$|x - c| \leq 1/32 \implies |r(x)| \leq |r(c)| + \sum_{j=1}^{15} \frac{|r^{(j)}(c)|}{j!32^j} < 33/20$$

and the values $r(27/4), r^{(j)}(27/4)$, $1 \leq j \leq 15$, verify Eq. (C.44) with interval rounding at 256 bits. The definition of $\tilde{r}$ gives boundedness and vanishing derivatives outside $[-7, 7]$. $\square$

Proof of Claim 7. To bound $|\mathcal{D}^4 \tilde{H}_i(t)|$ when $|P_i(t)| < 1$, define

$$\tilde{C}[a, b] := \sum_{|\beta|=b} (\mathbb{E}[\tilde{F}(W, X)\psi_a(W)\psi_\beta(X)])^2 \quad (a, b \geq 0).$$

For $\alpha \in \{0, 1, \dots\}^3$, write $\alpha! := \prod_i \alpha_i!$ and $\partial^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$; $[z^\alpha]$ denotes the coefficient of $\prod_i z_i^{\alpha_i}$. Derivatives at X are evaluated at $x = X$. Set

$$B_{\alpha, l}(x) := \frac{\alpha!}{l!} [z^\alpha] \left(\sum_{i=1}^3 \sum_{h=1}^4 \frac{\tilde{r}^{(h)}(x_i)}{h!} z_i^h \right)^l \quad (1 \leq l \leq |\alpha| \leq 4). \quad (\text{C.46})$$

For $1 \leq l \leq j \leq 4$, set

$$T_{j, l} := \sum_{|\alpha|=j} \frac{j!}{\alpha!} \mathbb{E}[B_{\alpha, l}(X)^2], \quad \begin{array}{c|cccc} j \backslash l & 1 & 2 & 3 & 4 \\ \hline 1 & 0.099 & & & \\ 2 & 0.221 & 0.335 & & \\ 3 & 0.36 & 3.38 & 4.88 & \\ 4 & 1.11 & 24.3 & 124 & 110 \end{array} \quad (\text{C.47})$$

Use $(n)_0 := 1$, $(n)_j := n(n-1)\cdots(n-j+1)$ for $1 \leq j \leq n$, $(n)_j := 0$ for $j > n \geq 0$, and $(-1)!! := 1$. We set $0^0 = 1$.

Claim 8. Each entry in Eq. (C.47) is a strict upper bound for $T_{j, l}$, and

$$\tilde{C}[a, b] \geq 0, \quad \sum_{a, b} \tilde{C}[a, b] = 1, \quad \tilde{H}_i(t) = \sum_{a, b} \tilde{C}[a, b] P_i(t)^a (-t)^b.$$

This series defines the extension of $\tilde{H}_i$ to complex $|t| \leq 1$. For nonnegative integers p, j and $0 \leq \rho < 1$,

$$\begin{aligned} \sum_{a \geq p, b \geq 0} (a)_p \rho^{a-p} \tilde{C}[a, b] &\leq \frac{2(2p-3)!!}{\pi(1-\rho^2)^{p-1/2}} \quad (1 \leq p \leq 4), \\ \sum_{a \geq p, b \geq j} (a)_p (b)_j \rho^{a-p} \tilde{C}[a, b] &\leq \left(\sum_{l=1}^j \sqrt{\frac{2T_{j,l}(2p+2l-3)!!}{\pi(1-\rho^2)^{p+l-1/2}}} \right)^2 \quad (j \geq 1, p+j \leq 4). \end{aligned} \quad (\text{C.48})$$

For $i \in \{0, 15, 16\}$, $|t| = 1$ and $|P_i(t)| < 1$,

$$\begin{aligned} |\mathcal{D}^4 \tilde{H}_i(t)| &\leq 24 \sum_{\substack{p, j \geq 0 \\ 1 \leq p+j \leq 4}} \frac{1}{p!j!} [z^4] \left(\sum_{h=1}^4 \frac{|\mathcal{D}^h P_i(t)|}{h!} z^h \right)^p \left(\sum_{h=1}^4 \frac{z^h}{h!} \right)^j \\ &\cdot \sum_{a \geq p, b \geq j} (a)_p (b)_j |P_i(t)|^{a-p} \tilde{C}[a, b]. \end{aligned} \quad (\text{C.49})$$

The series and its first four $\mathcal{D}$ -derivatives converge uniformly on $\{t : |t| \leq 1, |P_i(t)| \leq \rho\}$ for every $0 \leq \rho < 1$.

For $z \in \mathbb{C}$ with $-1 < \Re z < 1$, the Gaussian density in coordinates $u = (x+y)/\sqrt{2}$, $v = (x-y)/\sqrt{2}$ continues to

$$\frac{\exp(-u^2/(2(1+z)) - v^2/(2(1-z)))}{2\pi\sqrt{(1+z)(1-z)}}, \quad (\text{C.50})$$

with the analytic square-root branch positive for real $-1 < z < 1$. For $t = e^{i\theta}$, $0 < \theta < \pi$, and $\Re((1 \pm P_i(t))^{-1}) > 0$, multiply the four densities Eq. (C.50) at correlations $P_i(t), -t, -t, -t$. For $\sigma = \pm 1$, under the probability law whose density is the absolute value of this product divided by its integral over $\mathbb{R}^8$, set

$$\begin{aligned} Y_{0,\sigma} &:= \Re((1 + \sigma P_i(t))^{-1}) \frac{(W + \sigma W')^2}{2}, \\ Y_{1,\sigma} &:= \frac{1}{4} \sum_{j=1}^3 (X_j + \sigma X'_j)^2, \quad Y := (Y_{0,+}, Y_{0,-}, Y_{1,+}, Y_{1,-}). \end{aligned}$$

For $1 \leq k \leq 4$, define

$$\begin{aligned} \ell_k(Y) &:= -\frac{k!}{2} [\epsilon^k] \sum_{\sigma=\pm 1} \left(\ln(1 + \sigma P_i(t\epsilon)) + 3 \ln(1 - \sigma t\epsilon) \right. \\ &\quad \left. + \frac{Y_{0,\sigma}}{\Re((1 + \sigma P_i(t))^{-1})(1 + \sigma P_i(t\epsilon))} + \frac{2Y_{1,\sigma}}{1 - \sigma t\epsilon} \right), \end{aligned} \quad (\text{C.51})$$

holding Y and $\Re((1 + \sigma P_i(t))^{-1})$ constant in ϵ .

Claim 9. For $i \in \{0, 15, 16\}$ and every compact interval $I \subset (0, \pi)$ with $\inf_{\theta \in I} \Re((1 \pm P_i(e^{i\theta}))^{-1}) > 0$, $\tilde{H}_i$ continues analytically to a neighborhood of the arc. For $\theta \in I$ and $t = e^{i\theta}$,

$$|\mathcal{D}^4 \tilde{H}_i(e^{i\theta})| \leq \frac{(2/\sqrt{2 \sin \theta})^3 \sqrt{\mathbb{E} |\ell_1^4 + 6\ell_1^2 \ell_2 + 3\ell_2^2 + 4\ell_1 \ell_3 + \ell_4|^2}}{\sqrt{|1 - P_i(t)|^2 \Re((1 + P_i(t))^{-1}) \Re((1 - P_i(t))^{-1})}}. \quad (\text{C.52})$$

The coordinates of Y are independent, with

$$\mathbb{E} e^{sY_{0,\sigma}} = (1 - 2s)^{-1/2}, \quad \mathbb{E} e^{sY_{1,\sigma}} = (1 - 2s)^{-3/2} \quad (\sigma = \pm 1, s < 1/2).$$

Use the 6284 closed intervals

$$I \in \left\{ \left[\frac{k}{4000}, \frac{k+1}{4000} \right] : 0 \leq k \leq 6282 \right\} \cup \left\{ \left[\frac{6283}{4000}, \frac{\pi}{2} \right] \right\}.$$

The interval computations specified at the end of this subsection give, for every (i, I) , an upper bound

$$\sup_{\theta \in I} |\mathcal{D}^4 \tilde{H}_i(e^{i\theta})| \leq U_{i,I} < \infty,$$

with at least one of Claims 8 and 9 applicable throughout I . Before rounding to the displayed decimal bounds, the interval computations give

$$\begin{aligned} & \left(\frac{2}{\pi} \sum_I |I| \left(\sum_{i \in \{0,15,16\}} \lambda_i U_{i,I} \right)^2 \right)^{1/2} < 215825.836929, \\ & \frac{\left(\frac{2}{\pi} \sum_I |I| \left(\sum_{i \in \{0,15,16\}} \lambda_i U_{i,I} \right)^2 \right)^{1/2}}{\sqrt{14 \cdot 681^7}} + 4\sqrt{\Pr(F \neq \tilde{F})} \sum_{i \in \{0,15,16\}} \lambda_i < 0.000007875733380418. \end{aligned} \quad (\text{C.53})$$

Equation (4.2) and Claim 8 give

$$\sum_{m \geq 0} |[t^m] \tilde{H}_i| \leq \sum_{a,b} \tilde{C}[a,b] \left(\sum_{m \geq 1} |[t^m] P_i| \right)^a \leq 1,$$

so convergence is uniform on the closed disk. The normalized polynomials P_i specified by Table 1 satisfy $[t]P_i > 0$, $[t^3]P_i < 0$, $[t]P_i - [t^3]P_i = 1$, whence

$$|P_i(e^{i\theta})|^2 = 1 + 4([t]P_i)([t^3]P_i) \cos^2 \theta < 1 \quad (e^{i\theta} \neq \pm i), \quad P_i(i) = i, \quad \Re((1 \pm P_i(i))^{-1}) = \frac{1}{2}.$$

Claim 8 gives continuous derivatives through order four away from $\pm i$; Claim 9 and conjugation give analytic continuation near $\pm i$. Since

$$|P_i(t)| \leq ([t]P_i - [t^3]P_i)|t| = |t| < 1 \quad (|t| < 1),$$

the integral against the product Gaussian density is holomorphic on the open unit disk. It agrees there with the Hermite series by Eq. (3.3) on the real interval and the identity theorem; the arc integral continues this same function. Thus $\tilde{H}_i(e^{i\theta})$ is C^4 and $d^4/d\theta^4 = (i\mathcal{D})^4 = \mathcal{D}^4$. Oddness and real coefficients give $\tilde{H}_i(-t) = -\tilde{H}_i(t)$ and $\tilde{H}_i(\bar{t}) = \overline{\tilde{H}_i(t)}$, hence

$$\begin{aligned} \left\| \mathcal{D}^4 \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right\|_{L_2(\mathbb{T})}^2 &= \frac{2}{\pi} \int_0^{\pi/2} \left| \sum_{i \in \{0,15,16\}} \lambda_i \mathcal{D}^4 \tilde{H}_i(e^{i\theta}) \right|^2 d\theta \\ &\leq \frac{2}{\pi} \sum_I |I| \left(\sum_{i \in \{0,15,16\}} \lambda_i U_{i,I} \right)^2. \end{aligned} \quad (\text{C.54})$$

Thus Eq. (C.53) gives

$$\left\| \mathcal{D}^4 \sum_{i \in \{0,15,16\}} \lambda_i \tilde{H}_i \right\|_{L_2(\mathbb{T})} < 215825.836929 \quad (\text{C.55})$$

and Eq. (C.43). □

Proof of Claim 8. Eq. (C.48) implies Eq. (C.49) by the triangle inequality and the expansions

$$\begin{aligned}\mathcal{D}^4(P_i(t)^a(-t)^b) &= 24[z^4]P_i(te^z)^a(-t)^be^{bz}, \\ P_i(te^z)^a &= \sum_{p=0}^a \frac{(a)_p}{p!} P_i(t)^{a-p}(P_i(te^z) - P_i(t))^p, \\ P_i(te^z) - P_i(t) &= \sum_{h=1}^4 \frac{\mathcal{D}^h P_i(t)}{h!} z^h + O(z^5), \\ e^{bz} &= \sum_{j=0}^b \frac{(b)_j}{j!} (e^z - 1)^j, \quad \mathcal{D}^h P_i(t) = ([t]P_i)t + 3^h([t^3]P_i)t^3.\end{aligned}$$

For $|z| \leq 1$ and $1 \leq h \leq 4$,

$$|\mathcal{D}^h P_i(z)| \leq [t]P_i - 3^h[t^3]P_i.$$

The expansions and Eq. (C.48) therefore give, for $0 \leq j \leq 4$ and $0 \leq \rho < 1$,

$$\sum_{a,b} \tilde{C}[a,b] \sup_{\substack{|z| \leq 1 \\ |P_i(z)| \leq \rho}} \left| \mathcal{D}^j(P_i(t)^a(-t)^b) \Big|_{t=z} \right| < \infty,$$

proving uniform convergence, in particular near ± 1 since $|P_i(\pm 1)| < 1$.

To prove Eq. (C.48), conditioning on X gives

$$\tilde{C}[a,b] = \sum_{|\beta|=b} \left(\mathbb{E} \left[q_a \left(\sum_i \tilde{r}(X_i) \right) \psi_\beta(X) \right] \right)^2,$$

where q_a is defined in Eq. (C.13). Parseval and Eq. (3.3) give $\sum_{a,b} \tilde{C}[a,b] = 1$ and the expansion of $\tilde{H}_i$ in Claim 8. For nonnegative integers p, l with $p + l \geq 1$, $s \in \mathbb{R}$, and $0 \leq \rho < 1$, we use

$$\sum_{a \geq p} (a)_p \rho^{a-p} |q_a^{(l)}(s)|^2 \leq \frac{2(2p+2l-3)!!}{\pi(1-\rho^2)^{p+l-1/2}}. \quad (\text{C.56})$$

For $j = 0$, Parseval and Eq. (C.56) with $l = 0$ give

$$\begin{aligned}\sum_{a \geq p, b \geq 0} (a)_p \rho^{a-p} \tilde{C}[a,b] &= \mathbb{E} \left[\sum_{a \geq p} (a)_p \rho^{a-p} \left| q_a \left(\sum_i \tilde{r}(X_i) \right) \right|^2 \right] \\ &\leq \frac{2(2p-3)!!}{\pi(1-\rho^2)^{p-1/2}} \quad (1 \leq p \leq 4).\end{aligned}$$

For $a \geq 0$, $|\alpha| = j \in \{1, 2, 3, 4\}$, and $\beta \in \{0, 1, \dots\}^3$, Taylor expansion and Gaussian integration by parts using $(\phi \text{He}_n)' = -\phi \text{He}_{n+1}$ give

$$\begin{aligned}\partial^\alpha q_a \left(\sum_i \tilde{r}(x_i) \right) &= \sum_{l=1}^{|\alpha|} B_{\alpha,l}(x) q_a^{(l)} \left(\sum_i \tilde{r}(x_i) \right), \\ \mathbb{E} \left[\partial^\alpha q_a \left(\sum_i \tilde{r}(X_i) \right) \psi_\beta(X) \right] &= \sqrt{\frac{(\beta + \alpha)!}{\beta!}} \mathbb{E} \left[q_a \left(\sum_i \tilde{r}(X_i) \right) \psi_{\beta+\alpha}(X) \right], \\ \sum_{|\alpha|=j} \frac{j!}{\alpha!} \left\| \partial^\alpha q_a \left(\sum_i \tilde{r}(X_i) \right) \right\|_2^2 &= \sum_b (b)_j \tilde{C}[a,b],\end{aligned}$$

The last equality follows from Parseval and $\sum_{|\alpha|=j} (j!/\alpha!) \prod_i (\beta_i)_{\alpha_i} = (|\beta|)_j$. Boundedness of $\tilde{r}, \dots, \tilde{r}^{(4)}$ and Eq. (C.13) justify integration by parts. The triangle inequality gives

$$\left(\sum_{a \geq p, b \geq j} (a)_p (b)_j \rho^{a-p} \tilde{C}[a, b] \right)^{1/2} \leq \sum_{l=1}^j \left(T_{j,l} \sup_{s \in \mathbb{R}} \sum_{a \geq p} (a)_p \rho^{a-p} |q_a^{(l)}(s)|^2 \right)^{1/2}.$$

Apply Eq. (C.56); finite sums and monotone convergence justify summation over a .

For Eq. (C.56), with $u, v \in \mathbb{R}$ and $n = p + l - 1$, use

$$\sum_{a \geq p} (a)_p \rho^{a-p} q_a^{(l)}(u) q_a^{(l)}(v) = 4 \partial_u^n \partial_v^n \frac{e^{-(u^2 - 2\rho uv + v^2)/(2(1-\rho^2))}}{2\pi \sqrt{1-\rho^2}}. \quad (\text{C.57})$$

Fourier inversion, Cauchy–Schwarz, and the Gaussian moment formula give

$$\begin{aligned} 4 \left| \partial_u^n \partial_v^n \frac{e^{-(u^2 - 2\rho uv + v^2)/(2(1-\rho^2))}}{2\pi \sqrt{1-\rho^2}} \right| &\leq \frac{1}{\pi^2} \int_{\mathbb{R}^2} |\xi \eta|^n e^{-(\xi^2 + 2\rho \xi \eta + \eta^2)/2} d\xi d\eta \\ &\leq \frac{1}{\pi^2} \int_{\mathbb{R}^2} \xi^{2n} e^{-(\xi^2 + 2\rho \xi \eta + \eta^2)/2} d\xi d\eta = \frac{2(2n-1)!!}{\pi(1-\rho^2)^{n+1/2}}. \end{aligned}$$

Setting $u = v = s$ proves Eq. (C.56). To prove Eq. (C.57), use Eq. (C.13), $(\phi \text{He}_j)' = -\phi \text{He}_{j+1}$, and Mehler's identity:

$$\begin{aligned} \sum_{a \geq p} (a)_p \rho^{a-p} q_a^{(l)}(u) q_a^{(l)}(v) &= 4\phi(u)\phi(v) \sum_{k \geq 0} \frac{\rho^k}{k!} \text{He}_{k+n}(u) \text{He}_{k+n}(v) \\ &= 4\partial_u^n \partial_v^n \left(\phi(u)\phi(v) \sum_{k \geq 0} \frac{\rho^k}{k!} \text{He}_k(u) \text{He}_k(v) \right). \end{aligned}$$

At $\rho = 0$, only $a = p$ remains. For $0 < \rho < 1$ and arbitrary nonnegative integer shifts p, l , termwise differentiation is justified locally uniformly in $u, v \in \mathbb{R}$ by

$$\begin{aligned} &\sum_{k \geq 0} \frac{\rho^k}{k!} |\text{He}_{k+p}(u) \text{He}_{k+l}(v)| \\ &\leq \left(\sup_{k \geq 0} \frac{\sqrt{(k+p)!(k+l)!}}{k!} \rho^{k/2} \right) \left(\sum_{k \geq 0} \rho^{k/2} \psi_{k+p}(u)^2 \right)^{1/2} \left(\sum_{k \geq 0} \rho^{k/2} \psi_{k+l}(v)^2 \right)^{1/2}. \end{aligned}$$

For fixed p, l and $0 < \rho < 1$,

$$\frac{\sqrt{(k+p)!(k+l)!}}{k!} \rho^{k/2} \leq (k+p+l+1)^{(p+l)/2} \rho^{k/2} \longrightarrow 0 \quad (k \rightarrow \infty).$$

Mehler's identity with equal arguments gives

$$\sum_{k \geq 0} \rho^{k/2} \psi_{k+p}(u)^2 \leq \frac{\rho^{-p/2}}{\sqrt{1-\rho}} \exp\left(\frac{\sqrt{\rho} u^2}{1+\sqrt{\rho}}\right).$$

Apply the preceding sum inequality to tails. The displayed limit and these bounds for (p, u) and (l, v) give uniform convergence on compact sets of $u, v \in \mathbb{R}$.

Each monomial of $B_{\alpha,l}^2$ contains $2l \leq 8$ derivative factors. Independence factors its expectation into one-dimensional moments. For each such moment, with $k \in \{0, 1, \dots\}^4$ and $1 \leq |k| \leq 8$, parity and vanishing derivatives give

$$\begin{aligned} & \int_{\mathbb{R}} \prod_{h=1}^4 (\tilde{r}^{(h)}(x))^{k_h} \phi(x) dx \\ &= \left(1 + (-1)^{\sum_h (h+1)k_h}\right) \left(\int_0^{27/4} \prod_{h=1}^4 (r^{(h)}(x))^{k_h} \phi(x) dx + \int_0^{1/4} \prod_{h=1}^4 (\tilde{r}^{(h)}(27/4 + u))^{k_h} \phi(27/4 + u) du \right). \end{aligned}$$

The moment with $|k| = 0$ is 1. On each integration interval, $\deg \tilde{r}^{(h)} \leq 25$ for $1 \leq h \leq 4$, so each product has degree at most $25|k| \leq 200$. For $a \in \mathbb{R}$, $l \geq 0$, and integer $k \geq 1$, Gaussian integration gives

$$\begin{aligned} & \int_0^l \phi(a + u) du = \frac{1}{2} [\operatorname{erfc}(a/\sqrt{2}) - \operatorname{erfc}((a + l)/\sqrt{2})], \\ & \int_0^l u \phi(a + u) du = \phi(a) - \phi(a + l) - a \int_0^l \phi(a + u) du, \\ & \int_0^l u^{k+1} \phi(a + u) du = k \int_0^l u^{k-1} \phi(a + u) du - a \int_0^l u^k \phi(a + u) du - l^k \phi(a + l) \quad (k \geq 1). \end{aligned} \tag{C.58}$$

For $(a, l) = (0, 27/4), (27/4, 1/4)$, these recurrences and Eq. (C.41), evaluated with interval rounding at 1536 bits, give the strict bounds in Eq. (C.47). $\square$

Proof of Claim 9. For $\Re((1 \pm z)^{-1}) > 0$, we use

$$\int_{\mathbb{R}^2} \left| \frac{e^{-u^2/(2(1+z)) - v^2/(2(1-z))}}{2\pi\sqrt{(1+z)(1-z)}} \right| du dv = \left(|1 - z^2| \Re((1+z)^{-1}) \Re((1-z)^{-1}) \right)^{-1/2}, \tag{C.59}$$

and the coefficient identity

$$4![\epsilon^4] \exp\left(\sum_{k=1}^4 \frac{\ell_k(Y)}{k!} \epsilon^k\right) = \ell_1^4 + 6\ell_1^2 \ell_2 + 3\ell_2^2 + 4\ell_1 \ell_3 + \ell_4.$$

The right-hand side equals the fourth derivative in ϵ at $\epsilon = 0$ of the product density at correlations $P_i(te^\epsilon), -te^\epsilon, -te^\epsilon, -te^\epsilon$, divided by that product density at $\epsilon = 0$. The integration coordinates W, W', X, X' are held fixed. Since $|\tilde{F}| \leq 1$, Cauchy–Schwarz and

$$\Re((1 \pm e^{i\theta})^{-1}) = \frac{1}{2}, \quad |1 - e^{2i\theta}| = 2 \sin \theta \quad (0 < \theta < \pi)$$

give Eq. (C.52). On a sufficiently small bounded complex neighborhood of the compact arc, the closure satisfies

$$\begin{aligned} \Re((1 \pm P_i(t))^{-1}) &\geq \frac{1}{2} \inf_{\theta \in I} \Re((1 \pm P_i(e^{i\theta}))^{-1}) > 0, \\ \Re((1 \pm t)^{-1}) &\geq \frac{1}{4}. \end{aligned}$$

Each of the first four $\mathcal{D}$ -derivatives of the product density is that density times a polynomial of degree at most eight in W, W', X, X' , with coefficients bounded on this neighborhood. The density and these derivatives are therefore bounded by a constant multiple of

$$\begin{aligned} & \left(1 + W^2 + (W')^2 + \sum_{j=1}^3 (X_j^2 + (X'_j)^2) \right)^4 \\ & \cdot \exp \left[-\frac{1}{4} \min \left\{ \frac{1}{2}, \inf_{\substack{\theta \in I \\ \sigma = \pm 1}} \Re((1 + \sigma P_i(e^{i\theta}))^{-1}) \right\} \cdot \left(W^2 + (W')^2 + \sum_{j=1}^3 (X_j^2 + (X'_j)^2) \right) \right]. \end{aligned}$$

This integrable bound is independent of t in that neighborhood. Differentiation under the integral therefore gives the analytic continuation of $\tilde{H}_i$.

To verify Eq. (C.59), integrate the absolute density separately in u, v . After normalization,

$$\sqrt{\Re((1+z)^{-1})}u, \quad \sqrt{\Re((1-z)^{-1})}v$$

are independent standard Gaussians. Thus $Y_{0,\pm}$ are their individual squares and $Y_{1,\pm}$ sums of three such squares, proving the stated independence and moment-generating functions. The Gaussian change of variables is made at $\epsilon = 0$, after differentiation at fixed W, W', X, X' ; hence Y and $\Re((1 + \sigma P_i(t))^{-1})$ remain fixed as required in Eq. (C.51). The logarithm of the product density at correlations $P_i(te^\epsilon), -te^\epsilon, -te^\epsilon, -te^\epsilon$ gives Eq. (C.51). The expansions

$$\ln(1+u) = \sum_{h=1}^4 \frac{(-1)^{h+1}}{h} u^h + O(u^5), \quad (1+u)^{-1} = \sum_{h=0}^4 (-u)^h + O(u^5)$$

apply to the two arguments in Eq. (C.51) after factoring $1 + \sigma P_i(t)$ and $1 - \sigma t$, respectively, with

$$u = \frac{\sigma(P_i(te^\epsilon) - P_i(t))}{1 + \sigma P_i(t)} \quad \text{or} \quad u = \frac{-\sigma t(e^\epsilon - 1)}{1 - \sigma t}.$$

To evaluate the expectation in Eq. (C.52), in the rest of this proof set

$$\alpha = (1/2, 1/2, 3/2, 3/2), \quad (Y_1, Y_2, Y_3, Y_4) = (Y_{0,+}, Y_{0,-}, Y_{1,+}, Y_{1,-}).$$

Define

$$Q_{\alpha_h, k}(y) := \sum_{j=0}^k \binom{k}{j} (-2)^{k-j} \left(\prod_{l=j}^{k-1} (\alpha_h + l) \right) y^j \quad (1 \leq h \leq 4, k \geq 0),$$

with empty products one; $[y^k]Q_{\alpha_h, k}(y) = 1$. For nonnegative integers j, k, n , we use

$$\begin{aligned} \mathbb{E}[Q_{\alpha_h, k}(Y_h) Q_{\alpha_h, j}(Y_h)] &= \mathbf{1}_{k=j} 4^k k! \prod_{l=0}^{k-1} (\alpha_h + l), \\ Y_h^n &= \sum_{k=0}^n 2^{n-k} \binom{n}{k} \left(\prod_{l=k}^{n-1} (\alpha_h + l) \right) Q_{\alpha_h, k}(Y_h). \end{aligned} \tag{C.60}$$

For $k \in \{0, 1, \dots\}^4$, $|k| \leq 4$, set

$$p_k := \frac{\mathbb{E}[(\ell_1^4 + 6\ell_1^2\ell_2 + 3\ell_2^2 + 4\ell_1\ell_3 + \ell_4) \prod_{h=1}^4 Q_{\alpha_h, k_h}(Y_h)]}{\prod_{h=1}^4 (4^{k_h} k_h! \prod_{j=0}^{k_h-1} (\alpha_h + j))}.$$

Since $\deg_Y \ell_j \leq 1$ for $1 \leq j \leq 4$, independence and Eq. (C.60) give

$$\mathbb{E}|\ell_1^4 + 6\ell_1^2\ell_2 + 3\ell_2^2 + 4\ell_1\ell_3 + \ell_4|^2 = \sum_{|k| \leq 4} |p_k|^2 \prod_{h=1}^4 \left(4^{k_h} k_h! \prod_{j=0}^{k_h-1} (\alpha_h + j) \right). \tag{C.61}$$

For Eq. (C.60), use

$$\sum_{k \geq 0} Q_{\alpha_h, k}(y) \frac{z^k}{k!} = (1 + 2z)^{-\alpha_h} \exp\left(\frac{yz}{1 + 2z}\right) \quad (|z| < 1/2).$$

The moment-generating functions in Claim 9 give

$$\mathbb{E} \left[\frac{\exp \left(Y_h \left(\frac{u}{1+2u} + \frac{v}{1+2v} \right) \right)}{(1+2u)^{\alpha_h} (1+2v)^{\alpha_h}} \right] = (1-4uv)^{-\alpha_h}.$$

For $|u|, |v| < 1/8$, the integrand is bounded by $(4/3)^{2\alpha_h} e^{Y_h/3}$, with finite expectation $(4/3)^{2\alpha_h} 3^{\alpha_h}$. Taking $k!j![u^k v^j]$ under the expectation proves orthogonality. Substitution $z = s/(1-2s)$ gives

$$e^{sY_h} = \sum_{k \geq 0} \frac{Q_{\alpha_h, k}(Y_h) s^k}{k! (1-2s)^{\alpha_h + k}};$$

taking $n![s^n]$ proves the second identity. □

Verification of the interval bounds. For each (i, I) , evaluate Eq. (C.49) using Eq. (C.48) and Eq. (C.47) when $\sup_{\theta \in I} |P_i(e^{i\theta})| < 99/100$, and evaluate Eq. (C.52) when

$$\inf I \geq 3/5, \quad \inf_{\theta \in I} \Re((1 \pm P_i(e^{i\theta}))^{-1}) > 0.$$

Use interval rounding at 192 bits on the entire closed interval. For every (i, I) , at least one estimate satisfying the displayed hypotheses gives a finite upper endpoint; $U_{i, I}$ is the minimum of the available upper endpoints. The sum in Eq. (C.54) and the interval bound from Eq. (C.45), before rounding to the displayed decimal bound, give Eq. (C.53).

C.5 Verification of the numerical bounds

From the repository root, run

```
python verify.py --upper
```

The inputs and source are in `upper/`. Source paths are relative to `upper/source/certification/`, except `verify.py`, `data_link.py`, `requirements.txt`, and paths beginning with `data/` or `upper/`, which are relative to the repository root. Generated mathematical data files use the same relative paths under `runs/complete-*/upper_results/run-*/work/certification/`. The tables give the arithmetic precision in bits and the bounds verified. SHA-256 values are computed from file contents and compared with the recorded values.

Source	Bits	Checks and uses
<code>data_link.py</code>	Exact	Checks SHA-256 values and exact equality with the supplied rounding data and data defining F_* ; Eq. (4.2).
<code>weighted_staircase/freeze_recipe.py</code>	Exact	Inserts F_* for $i = 14$; <code>upper/verify.py</code> checks the SHA-256 value of <code>weighted_staircase/explicit_staircase_mixture.json</code> .
<code>upper_head_681/full_head_plan.py</code>	512	Quadrature and omitted-integral bounds in Claim 2; output entries <code>tilted_J0_bound</code> , <code>tilted_J2_bound</code> for the two exponential moments.
<code>upper_head_681/certified_b_matrix.py</code>	512	Quadrature nodes and weights, values of ψ_j , $\hat{B}$, $\hat{U}$; input and summation errors; output entry <code>A_Hilbert_error</code> for the combined Hermite coefficient error $< 4.096939258130 \cdot 10^{-10}$.
<code>upper_head_681/contract_certified_head.py</code>	256; binary64	Bounds on sine and cosine values; polynomial cubes and weighted sums in Eq. (C.15).
<code>upper_head_681/verify_plane_budget.py</code>	256	Errors in the discrete Fourier transforms, weighted sums, and omission of $\hat{U}_{av} < 10^{-18}$ in Claim 3; Eq. (C.10).
<code>compose_head.py</code>	256	Coefficients through degree 681 in Eq. (C.11), with Eq. (C.9) subtracted; Eq. (C.5).

The files `upper_rounding.json` and `staircase.json` in `data/` are checked for the indices $0, \dots, 31$ and their specified rounding functions, positive weights, 2048 strips of width $1/256$ covering $[0, 8)$, increasing $r_{k,a}$, and signs in $\{-1, 1\}$. For $0 \leq k < 2048$, row k , numbered from 0, of `data/staircase.json` stores $r_{k,a}$, numbered from 1, in its `breakpoints` list, with signs $1, -1, 1, \dots$. These are the rational breakpoints specified below.

The accepted intervals in step 2 are the `inner_cells` entries of `upper_head_681/full_head_quadrature_plan.json`. The routine `upper_head_681/contract_certified_head.py`, with inputs from `upper_head_681/certified_b_matrix.py`, generates `upper_head_681/certified_plane.npy`. Its entries are the rational array $\hat{C}$ used in Eq. (C.11) and Eq. (C.39); the error checks and both coefficient calculations use this same generated array.

The additional scalar check in Eq. (C.22), run automatically by `verify.py`, is

```
from flint import arb, ctx
ctx.prec = 256
assert (64 * (-arb(512)).exp() * arb(32)**681
        / (345 * arb.pi() * (343 * arb.fac_ui(681)).sqrt())
        < arb(3) / 10**18)
```

Source	Bits	Checks and uses
weighted_staircase/certify_head.py	384	Eq. (C.32), Eq. (C.31).
degree681/certify_linear_tails.py	256	Eq. (C.36), Eq. (C.37).
weighted_staircase/certify_nonadditive.py	384	Eq. (C.30); coefficients of $\lambda_{14}H_{14} + \sum_{i=17}^{31} \lambda_i H_i$ for <code>compose_head.py</code> .
certify_rare_additive_tails.py	256	Eq. (C.39) with the same $\hat{C}$ and errors; Eq. (C.38).
upper_tail/certify_quintic_continuation.py	256	Eq. (C.44), Eq. (C.42).
upper_tail/certify_factorized_energies.py	1536	Eq. (C.46), Eq. (C.58); the table in Eq. (C.47).
upper_head/arc/certify_complex_arc.py, bound	192	Eq. (C.51), Eq. (C.60), all 70 terms of Eq. (C.61); Eq. (C.52).
upper_tail/certify_heavy_tail_fine.py	192	Bounds Eq. (C.49), Eq. (C.52) on the 6284 intervals; their union $[0, \pi/2]$; Eq. (C.54), Eq. (C.55), Eq. (C.42).
certify_upper.py	256	Checks SHA-256 values, degrees, and all rounding indices; combines Eq. (C.5), Eq. (C.30), Eq. (C.38), and Eq. (C.40) before decimal rounding, including division by $\sqrt{14 \cdot 681^7}$ and Eq. (C.42).
verify.py	256	Bounds Eq. (C.5), Eq. (C.30), Eq. (C.38), Eq. (C.40), and $c_1 - \sum_{m \geq 2} c_m > 1000000/1779893$.

Arb is the library used for real and complex interval arithmetic; interval endpoints are rounded down for lower bounds and up for upper bounds. Binary64 calculations use round-to-nearest and the bounds of Claim 3; other finite calculations use exact rationals or these interval bounds. Verification assumes correct execution of these arithmetic operations and elementary functions. Use Python 3.12 or a compatible newer version, without `-O`, and `requirements.txt`. The program `upper/verify.py` assembles `weighted_staircase/explicit_staircase_mixture.json` from the stored inputs and runs all 12 numerical calculations in a fresh directory before writing `VERIFIED.json`. The program `verify.py` writes `ACCEPTED.json` only after the input comparisons, the scalar check above, and the five numerical inequalities in its table entry have been checked. The two acceptance reports are located at `runs/complete-*/upper_results/run-*/VERIFIED.json` and `runs/complete-*/ACCEPTED.json`, respectively. The option `-all -workers 4` also runs the lower certificate and requires a C++17 compiler and GMP/MPFR; `-data-only` checks input identities and rational comparisons without executing either numerical proof.

Staircase breakpoints. For each $k \in \{0, \dots, 2047\}$, substitute $y = (2k + 1)/512$ in Eq. (4.5). Round its real roots to the nearest multiples of 10^{-12} and list them in increasing order. For each triple $(k, a, +1)$ or $(k, a, -1)$ below, respectively add or subtract 10^{-12} from the a -th rounded root for this k . Leave all other rounded roots unchanged. Let $r_{k,a}$ denote the resulting breakpoints. All roots are simple, rounding is unique, and $r_{k,a} < r_{k,a+1}$ after these corrections.

k	a	± 1	k	a	± 1	k	a	± 1	k	a	± 1
44	1	-1	487	1	-1	948	5	+1	1261	3	+1
64	5	+1	619	5	-1	984	2	+1	1281	3	+1
181	5	-1	626	1	-1	1019	1	-1	1309	2	-1
286	1	-1	669	1	+1	1019	5	-1	1319	2	-1
305	5	+1	684	4	+1	1030	2	+1	1412	1	+1
342	1	+1	715	3	+1	1053	2	+1	1417	2	-1
369	1	-1	739	1	-1	1055	1	-1	1677	1	+1
402	2	-1	862	5	+1	1113	1	-1			
449	1	-1	902	5	+1	1125	1	-1			