SHARPER BOUNDS FOR THE COMPLEX GROTHENDIECK CONSTANT

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ABSTRACT. We show that $1.4 < K_G^{\mathbb{C}} < 1.404898554746$, where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant. The upper bound improves on Haagerup's bound of $1.40490913\ldots$ from 1987, and the lower bound improves on Davie's bound of 1.33807 from 1984. The upper bound combines ideas from the real and noncommutative complex cases of Grothendieck's inequality.

1. INTRODUCTION

The complex Grothendieck constant $K_G^{\mathbb{C}}$ is defined [G53] as the infimum over all $K \in (0,\infty)$ such that, for any $m, n \in \mathbb{N}$ and for any $m \times n$ complex matrix a_{ij}, we have

$$\sup_{\substack{x_1,\ldots,x_m,y_1,\ldots,y_n \in \mathbb{C}^{m+n-1}:\\ \|x_1\|=\cdots=\|x_m\|=\|y_1\|=\cdots=\|y_n\|=1}} \left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\,\langle x_i, y_j\rangle\right| \leq K \cdot \sup_{\substack{\varepsilon_1,\ldots,\varepsilon_m,\delta_1,\ldots,\delta_n \in \mathbb{C}:\\ |\varepsilon_1|=\cdots=|\varepsilon_m|=|\delta_1|=\cdots=|\delta_n|=1}} \left|\sum_{i,j} a_{ij}\varepsilon_i\overline{\delta_j}\right|. \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard complex inner product, which is linear in its first argument, and $\|\cdot\|$ denotes the standard Euclidean norm.

Besides being a fundamental quantity in functional analysis [Pis12, V74, GR18, BBLV12] and quantum information [BBLV12], the complex Grothendieck constant also has algorithmic applications. Tropp [Tro09] uses Grothendieck factorization (i.e. (1.1) in an equivalent form) in a randomized polynomial-time algorithm that, given a matrix with unit-norm columns, selects a large column submatrix with bounded condition number. For a complex matrix A commuting with a transitive group of permutation matrices, [BBLM20, Theorem 3.1] gives

$$\|A\|_{L_\infty\to L_1} \leq \|A\|_{L_2\to L_2} \leq K_G^{\mathbb{C}}\|A\|_{L_\infty\to L_1},$$

where the L_p norms use normalized counting measure. Thus the largest singular value approximates the $L_\infty \to L_1$ norm within a factor $K_G^{\mathbb{C}}$.

Haagerup proved $K_G^{\mathbb{C}} \leq \gamma_H^{-1} \overset{(2.6)}{\approx} 1.40490913\ldots$ in 1987 [Haa87], which, until now, remained the smallest upper bound on $K_G^{\mathbb{C}}$. Following improved upper bounds on $K_G^{\mathbb{C}}$ by [Kai73, Pis78], Haagerup used a complex analogue of Krivine's method for proving the bound $K_G^{\mathbb{R}} \leq \frac{\pi}{2\log(1+\sqrt{2})}$ [Kri77] on Grothendieck's real constant (which has since been improved in a sequence of works [BMMN13, Hei26, LSX+26, SLX+26]). In particular, [Haa87] first preprocesses the vectors x_i, y_j, mapping them to a tensor product (Fock) space (see

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(3.8)), where the map's coefficients are determined by the Taylor coefficients of the inverse of the function h from (2.3), i.e. $h(t)$ is the correlation of the phase of two standard jointly complex Gaussians with correlation $t \in (-1, 1)$. Then, the resulting vectors are projected onto complex Gaussians and the phase function $z \mapsto z/|z|$ is applied (if $z \neq 0$), producing ε_i, δ_j. The preprocessing is set up so that it cancels the nonlinearity that results from the Gaussian projection. This proof of [Haa87] is presented in Remark 3.3 and Lemma 3.2 below.

The lower bound $K_G^{\mathbb{C}} \geq 1.33807$ was shown in [D84] by computing the $\infty \to 1$ norm of $P_1 - \lambda I$, where P_1 is the projection onto the first degree Hermite-Fourier coefficients, and $\lambda \in \mathbb{R}$ is chosen to optimize this norm. The lower bound was independently improved [GFL26] to $K_G^{\mathbb{C}} > 1.35584631827168$ by subtracting odd Hermite projections from the identity map, as opposed to subtracting odd Hermite projections from $P_1 - \lambda I$, as was done in the real case [Hei26b, JM26]. Our main result is the following.

Theorem 1.1.

$$\boxed{1.4 < K_G^{\mathbb{C}} < 1.404898554746 < \gamma_H^{-1} - 10^{-5}.} \tag{1.2}$$

The proof of the upper bound of Theorem 1.1 synthesizes ideas for rounding schemes of the real Grothendieck inequality [BMMN13, Hei26, LSX+26, SLX+26] and also from the noncommutative complex case [NRV14].

In particular, we use a two-stage preprocessing with a mixed rounding scheme. First, a common preprocessor is applied to the vectors x_i, y_j in (1.1). Then, if the outcome of rolling a particular nonuniform six-sided die is not 6, the vectors x_i, y_j are further preprocessed in one of five different ways, four of which are small perturbations of Haagerup's preprocessing. Then, these vectors are projected onto complex Gaussians and the phase function is applied to output ε_i, δ_j. If the die outcome is 6, then no further preprocessing occurs, the vectors are projected onto complex Gaussians, and perturbed phase functions F, G from (4.6) are applied to them (or their conjugate reflections $z \mapsto (\overline{F(\overline{z})}, \overline{G(\overline{z})})$, each with 1/2 probability).

The first five die outcomes correspond to a "mixed rounding scheme," as used in [BMMN13, NR14, LSX+26, SLX+26]. The sixth outcome is a relative of the rounding scheme from [NRV14]. The novelty in this construction is both the two-stage process of the rounding scheme, and the combination of different preprocessing functions. For comparison, the mixed rounding schemes from [BMMN13, NR14, LSX+26, SLX+26] use a single pair of preprocessing maps.

If instead of the upper bound of Theorem 1.1, the reader is satisfied by the weaker statement $K_G^{\mathbb{C}} < \gamma_H^{-1}$, then a simpler proof than that used for Theorem 1.1 suffices. That simpler rounding scheme is analogous to the one described above, where the first five alternatives collapse to one case, so there are only two alternatives instead of six. Since the statement $K_G^{\mathbb{C}} < \gamma_H^{-1}$ is weaker than Theorem 1.1, we omit the details. Due to the different preprocessing steps in this two-stage mixed rounding scheme, the statement $K_G^{\mathbb{C}} < \gamma_H^{-1}$ does not follow from the most straightforward adaptation of the argument of [BMMN13].

The lower bound of Theorem 1.1 follows by upper bounding the $\infty \to 1$ norm of

$$T_n := P_{1,0} - \frac{37}{50} P_{2,1},$$

where $P_{p,q}$ is the orthogonal projection onto the complex Hermite component of bidegree (p, q) under standard Gaussian measure on $\mathbb{C}^n$. We prove in Theorem 7.1 that

$$\sup_{n \geq 1} \|T_n\|_{\infty \to 1} < \frac{2857}{4000} < \frac{5}{7},$$

which implies $K_G^{\mathbb{C}} > 4000/2857 > 1.4$.

Surprisingly, we were unable to successfully apply the "limiting Krivine schemes" directly from [SLX+26] in Theorem 1.1, although those schemes achieve the best bounds on the real constant $K_G^{\mathbb{R}}$. There it is shown that certain nonlinear maps can be applied to the correlation parameter. Perhaps these ideas could be incorporated somehow to improve (1.2).

Both the upper and lower bounds in Theorem 1.1 perform analytical reductions to inequalities which are certified numerically. Supporting codes for Theorem 1.1 are given at:

https://github.com/sheilman77/grothendieck_cx

1.1. **A Remaining Conjecture.** Haagerup [Haa87] conjectured that a full complex analogue of Krivine's argument should imply that $K_G^{\mathbb{C}}$ is equal to

$$\gamma_\dagger^{-1} = 1.404575934663742\ldots, \qquad \gamma_\dagger := \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1+\sin^2 t}}\, dt, \tag{1.3}$$

(see [Pis12, Section 4]), but this remains a conjecture.

2. Preliminaries on Haagerup's Approach

A standard complex Gaussian has density $w \mapsto \pi^{-1} e^{-|w|^2}$, $w \in \mathbb{C}$, so its second absolute moment is one. Let (X, Y) be a jointly proper standard complex Gaussian pair with $\mathbb{E}[X\overline{Y}] = z$, $|z| \le 1$. Haagerup's phase identity [Haa87, Lemma 3.2] (see also [FLZ18, Lemma 2.1]) is

$$\mathbb{E}\big[\mathrm{phase}(X)\overline{\mathrm{phase}(Y)}\big] = \mathfrak{h}(z) := \frac{\pi}{4} z \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; |z|^2\right) = \sum_{n\ge 0} h_n z |z|^{2n}, \tag{2.1}$$

where $\mathrm{phase}(w) = w/|w|$ for all $w \in \mathbb{C} \setminus \{0\}$ with $\mathrm{phase}(0) := 1$, and

$$h_k = \frac{\pi}{4(k+1)} \frac{\binom{2k}{k}^2}{16^k}, \qquad \frac{h_{k+1}}{h_k} = \frac{(2k+1)^2}{4(k+1)(k+2)}, \qquad \sum_{k\ge 0} h_k = 1. \tag{2.2}$$

Write $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. For any real $t \in (-1, 1)$, let

$$h(t) := \frac{\pi}{4} t \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; t^2\right) = \sum_{k\ge 0} h_k t^{2k+1}. \tag{2.3}$$

Thus h is an analytic odd series, whereas $\mathfrak{h}$ is the physical charge-one radial kernel. Here "charge one" means $K(e^{i\theta} z) = e^{i\theta} K(z)$ for all $\theta \in \mathbb{R}$ and $z \in \overline{\mathbb{D}}$. The integral form in [Haa87, Lemma 3.2] also yields

$$\mathfrak{h}(z) = z \int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{1 - |z|^2 \sin^2\theta}}\, d\theta, \qquad \forall\, z \in \overline{\mathbb{D}}. \tag{2.4}$$

Let $x_H \in (0,1)$ be the unique solution of

$$2x_H \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; x_H^2\right) = 1 + x_H, \tag{2.5}$$

and put

$$\gamma_H := \frac{\pi(1 + x_H)}{8} \approx 0.711789806684998. \tag{2.6}$$

(Recall $h(0) = 0$ by (2.3) and $h(1) = 1$ from (2.2), so the function $\phi(x) := h(x) - (\pi/8)(1+x)$ is negative at $x = 0$ and positive at $x = 1$, so there exists at least one x satisfying (2.5). To see uniqueness, recall that $h_0 = \pi/4$ and $h_k > 0$ for all $k \geq 1$, so $\phi'(x) > 0$ for all $0 < x < 1$.)

Let h^{-1} be the odd analytic inverse germ of (2.3), and let

$$\rho_*(t) := h^{-1}(\gamma_H t). \tag{2.7}$$

3. General Results on Kernels

We first isolate general facts in Haagerup's rounding procedure in terms of kernels. Let $S(E)$ denote the unit sphere of a complex Hilbert space E.

Definition 3.1. A map $\rho\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is an **allowable preprocessor** if, for every complex Hilbert space E, there are a complex Hilbert space F and maps $U, V\colon S(E) \to S(F)$ where

$$\langle U(x), V(y)\rangle = \rho\big(\langle x, y\rangle\big) \qquad \forall\, x, y \in S(E).$$

A function $K : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a **valid rounding kernel** if, $\forall$ E, $\forall$ $m, n \in \mathbb{N}$ and $\forall$ $x_1, \ldots, x_m$, $y_1, \ldots, y_n \in S(E)$, there are random variables $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \in S(\mathbb{C})$ such that

$$\mathbb{E}[\varepsilon_i \overline{\delta_j}] = K\big(\langle x_i, y_j\rangle\big), \qquad \forall\, 1 \leq i \leq m,\ 1 \leq j \leq n. \tag{3.1}$$

Lemma 3.2 (Kernel Bound)**.** *Let $r > 0$. Assume $K(z) = rz$ is valid. Then $K_G^{\mathbb{C}} \leq r^{-1}$.*

Proof. Choose unit vectors $x_1, \ldots, x_m, y_1, \ldots, y_n$ attaining the supremum on the left side of (1.1). Since $K(z) = rz$ is a valid kernel,

$$\begin{aligned}\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle x_i, y_j\rangle\right| &= \frac{1}{r}\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} K(\langle x_i, y_j\rangle)\right| \\ &\overset{(3.1)}{=} \frac{1}{r}\left|\mathbb{E}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\varepsilon_i\overline{\delta_j}\right| \leq \frac{1}{r}\sup_{\varepsilon_1,\ldots,\varepsilon_m,\delta_1,\ldots,\delta_n \in S(\mathbb{C})}\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\varepsilon_i\overline{\delta_j}\right|.\end{aligned}$$

□

We use the real odd Wiener space

$$\mathcal{A} = \left\{ u(t) = \sum_{k\geq 0} c_k t^{2k+1} : c_k \in \mathbb{R},\ \|u\|_{\mathcal{A}} := \sum_k |c_k| < \infty \right\}.$$

The radial extension of $u \in \mathcal{A}$ is

$$u^{\mathrm{phys}}(z) := \sum_{k\geq 0} c_k z |z|^{2k}. \tag{3.2}$$

Coefficient convolution and the triangle inequality give

$$\|u^{2j+1}\|_{\mathcal{A}} \leq \left(\sum_{k\geq 0}|c_k|\right)^{2j+1} = \|u\|_{\mathcal{A}}^{2j+1}, \tag{3.3}$$

for all $j \geq 0$. In particular, if $\|u\|_{\mathcal{A}} \leq 1$, then

$$\sum_{k\geq 0} h_k \|u^{2k+1}\|_{\mathcal{A}} \leq \sum_{k\geq 0} h_k \|u\|_{\mathcal{A}}^{2k+1} \leq \sum_{k\geq 0} h_k = 1. \tag{3.4}$$

Thus $h \circ u$ converges absolutely in $\mathcal{A}$. Expansion shows that its radial extension is $\mathfrak{h} \circ u^{\mathrm{phys}}$.

Remark 3.3. Recall $\gamma_H \approx 0.71178980668\ldots$ from (2.6). Let $\rho_*(t) := h^{-1}(\gamma_H t)$ for all $t \in [0,1]$. Then [Haa87] shows that $\|\rho_*\|_{\mathcal{A}} = 1$, so $K_{\rho_*}(z) = \gamma_H \cdot z$ is valid by Lemma 3.5 below, so Lemma 3.2 implies that $K_G^{\mathbb{C}} \le \gamma_H^{-1}$, recovering the main result of [Haa87]. To justify $\|\rho_*\|_{\mathcal{A}} = 1$, write $\rho_*(t) = \sum_{n\ge 0} c_n t^{2n+1}$. The known expansion of h^{-1} shows $c_0 = (1+x_H)/2$, $c_1 < 0$, $c_2 = 0$, and $c_n < 0$ for every $n > 2$. Since $\rho_*(1) = h^{-1}(\gamma_H) = x_H$ by (2.5),

$$\|\rho_*\|_{\mathcal{A}} = c_0 - \sum_{n\ge 1} c_n = 2c_0 - \sum_{n\ge 0} c_n = 2c_0 - x_H \overset{(2.5)}{=} 1, \qquad h(\rho_*(t)) = \gamma_H t. \tag{3.5}$$

Remark 3.4. The sign pattern of the coefficients of h^{-1} here is different from the analogous sign pattern for real scalars. In the latter setting, the inverse of the Gaussian correlation of two sign functions is a sine function, whose coefficients have alternating signs. But $c_n < 0$ for all $n > 2$, so bounds for $K_G^{\mathbb{R}}$ do not carry over directly to the complex case.

Lemma 3.5 (Existence and Convexity of Valid Kernels)**.** *If $\rho(t) = \sum_{k\ge 0} c_k t^{2k+1} \in \mathcal{A}$ and $\|\rho\|_{\mathcal{A}} \le 1$, then ρ^{phys} is an allowable preprocessor. For any $z \in \overline{\mathbb{D}}$ define*

$$K_\rho(z) := \mathfrak{h}(\rho^{\mathrm{phys}}(z)). \tag{3.6}$$

Then K_ρ is a valid rounding correlation kernel, with coefficient representative $h \circ \rho \in \mathcal{A}$.

More generally, let $F, G\colon \mathbb{C} \to S(\mathbb{C})$ be measurable. For a jointly proper standard circular complex Gaussian pair (X_z, Y_z) satisfying $\mathbb{E}[X_z \overline{Y_z}] = z$ for all $z \in \overline{\mathbb{D}}$, the function

$$K_{F,G}(z) := \mathbb{E}\big[F(X_z)\overline{G(Y_z)}\big], \qquad \forall\, z \in \overline{\mathbb{D}}, \tag{3.7}$$

is a valid complex correlation rounding kernel.

Convex combinations of valid rounding kernels are valid rounding kernels.

Proof. Put $s_\rho := \sum_{k\ge 0} |c_k|$. Note that $s_\rho \le 1$ by the assumption $\|\rho\|_{\mathcal{A}} \le 1$. After placing the tensor summands in mutually orthogonal Hilbert spaces, fix unit vectors $e_L, e_R \in H$ that are orthogonal to every tensor summand and to each other, and set

$$\begin{aligned} U(x) &:= \bigoplus_{k\ge 0} |c_k|^{1/2} (x^{\otimes(k+1)} \otimes \overline{x}^{\otimes k}) \oplus \Big((1-s_\rho)^{1/2}\, e_L\Big), \\ V(y) &:= \bigoplus_{k\ge 0} \mathrm{sign}(c_k) |c_k|^{1/2} (y^{\otimes(k+1)} \otimes \overline{y}^{\otimes k}) \oplus \Big((1-s_\rho)^{1/2}\, e_R\Big). \end{aligned} \tag{3.8}$$

Then $\|U(x)\| = \|V(y)\| = 1$ by definition of s_ρ, and

$$\langle U(x), V(y)\rangle \overset{(3.8)}{=} \sum_{k\ge 0} c_k \, \langle x, y\rangle^{k+1}\, \overline{\langle x, y\rangle}^{\,k} \overset{(3.2)}{=} \rho^{\mathrm{phys}}(\langle x, y\rangle). \tag{3.9}$$

Thus the maps U, V are feature maps witnessing that ρ^{phys} is an allowable preprocessor.

Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in S(H)$. Let $E := \mathrm{span}_{\mathbb{C}}\{U(x_1), \ldots, U(x_m), V(y_1), \ldots, V(y_n)\}$. Then E is a linear subspace with dimension at most $m+n$. Let g be a complex standard Gaussian in E. For every $v, w \in E$, our convention that the inner product is linear in its first argument gives

$$\mathbb{E}\Big[\langle v, g\rangle\, \overline{\langle w, g\rangle}\Big] = \langle v, w\rangle, \qquad \mathbb{E}[\langle v, g\rangle \langle w, g\rangle] = 0. \tag{3.10}$$

Consequently,

$$Z_i := \langle U(x_i), g\rangle, \qquad W_j := \langle V(y_j), g\rangle, \qquad \forall\, 1 \le i \le m,\ 1 \le j \le n,$$

form a jointly proper family of standard circular complex Gaussians, and (3.9) yields

$$\mathbb{E}[Z_i\overline{W_j}] \overset{(3.10)}{=} \langle U(x_i), V(y_j)\rangle \overset{(3.9)}{=} \rho^{\mathrm{phys}}(\langle x_i, y_j\rangle), \qquad \forall\, 1 \le i \le m,\ 1 \le j \le n. \tag{3.11}$$

Thus, on setting $\varepsilon_i := \mathrm{phase}(Z_i)$ and $\delta_j := \mathrm{phase}(W_j)$, identity (2.1) gives

$$\mathbb{E}[\varepsilon_i\overline{\delta_j}] \overset{(2.1)\wedge(3.11)}{=} \mathfrak{h}\big(\rho^{\mathrm{phys}}(\langle x_i, y_j\rangle)\big) \overset{(3.6)}{=} K_\rho(\langle x_i, y_j\rangle).$$

This proves the required validity for every finite instance.

It remains to verify the assertion for $K_{F,G}$. Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in S(H)$, and let $E' := \mathrm{span}_{\mathbb{C}}\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$. Then $\dim E' \le m + n$. Choose a standard proper circular complex Gaussian vector g' on E', and define the shared Gaussian projections

$$X_i := \langle x_i, g'\rangle, \qquad Y_j := \langle y_j, g'\rangle, \qquad \forall\, 1 \le i \le m,\ 1 \le j \le n.$$

The entire family $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ is jointly proper complex Gaussian. Moreover,

$$\mathbb{E}|X_i|^2 = \mathbb{E}|Y_j|^2 = 1, \qquad \mathbb{E}[X_i\overline{Y_j}] = \langle x_i, y_j\rangle, \qquad \forall\, 1 \le i \le m,\ 1 \le j \le n.$$

Thus, for each pair (i,j), the joint distribution of (X_i, Y_j) is the distribution of a jointly proper standard circular complex Gaussian pair (X_z, Y_z) with $z = \langle x_i, y_j\rangle$. Here we use that a centered proper complex Gaussian pair is determined by its covariance matrix. Now set

$$\varepsilon_i := F(X_i), \qquad \delta_j := G(Y_j). \tag{3.12}$$

Since F and G take values in $S(\mathbb{C})$, we have $|\varepsilon_i| = |\delta_j| = 1$ for all $1 \le i \le m$, $1 \le j \le n$, and

$$\mathbb{E}[\varepsilon_i\overline{\delta_j}] \overset{(3.12)}{=} \mathbb{E}\Big[F(X_i)\overline{G(Y_j)}\Big] \overset{(3.7)}{=} K_{F,G}(\langle x_i, y_j\rangle).$$

The use of one shared Gaussian vector g' is important: it places all the variables $\varepsilon_1, \ldots, \varepsilon_m$, $\delta_1, \ldots, \delta_n$ on a single probability space. The definition of a valid kernel requires such a joint realization, rather than a separate Gaussian pair for each (i,j). Hence $K_{F,G}$ is a valid complex rounding correlation kernel.

Finally, validity is preserved under convex combinations. Indeed, let $K^{(1)}, \ldots, K^{(r)}$ be valid kernels and let $\lambda_1, \ldots, \lambda_r \ge 0$ satisfy $\sum_{a=1}^r \lambda_a = 1$. For the fixed input $x_1, \ldots, x_m, y_1, \ldots, y_n \in S(H)$, choose for each $1 \le a \le r$ a joint unimodular family

$$\varepsilon_1^{(a)}, \ldots, \varepsilon_m^{(a)}, \delta_1^{(a)}, \ldots, \delta_n^{(a)} \in S(\mathbb{C}),$$

witnessing the validity of $K^{(a)}$. Let $A \in \{1, \ldots, r\}$ be an independent random selector with $\mathbb{P}(A = a) = \lambda_a$, for all $1 \le a \le r$. Define

$$\varepsilon_i := \varepsilon_i^{(A)}, \qquad \delta_j := \delta_j^{(A)}, \qquad \forall\, 1 \le i \le m,\ 1 \le j \le n.$$

Conditioning on A gives

$$\mathbb{E}[\varepsilon_i\overline{\delta_j}] = \mathbb{E}(\mathbb{E}[\varepsilon_i\overline{\delta_j}|A]) = \sum_{a=1}^r \lambda_a\,\mathbb{E}\Big[\varepsilon_i^{(a)}\overline{\delta_j^{(a)}}\Big] = \sum_{a=1}^r \lambda_a K^{(a)}(\langle x_i, y_j\rangle), \quad \forall\, 1 \le i \le m,\ 1 \le j \le n.$$

Therefore $\sum_{a=1}^r \lambda_a K^{(a)}$ is a valid rounding kernel.

□

Lemma 3.6 (Repairing a nonlinear kernel)**.** *Let $r \in \mathcal{A}$ with no degree one term. Let*

$$q(t) := at + r(t) \in \mathcal{A}, \qquad a > 0, \qquad \|r\|_{\mathcal{A}} \le \delta < a.$$

For all $0 \le \Gamma \le a-\delta$, the inverse germ $\eta(t) = q^{-1}(\Gamma t)$ belongs to $\mathcal{A}$ and $\|\eta\|_{\mathcal{A}} \le 1$. Moreover, if q represents a valid charge-one kernel, then inverse preprocessing by η^{phys} produces the linear valid kernel $z \mapsto \Gamma z$.

Proof. Write $r(t) = \sum_{m\ge1} r_m t^{2m+1}$ and $r^{\#}(t) := \sum_{m\ge1} |r_m| t^{2m+1}$. Starting with $v_0 = 0$, define formal power series v_{k+1} with nonnegative coefficients by

$$v_{k+1} := \frac{\Gamma t + r^{\#}(v_k)}{a}.$$

Since $r^{\#}$ has nonnegative coefficients, $v_k \le v_{k+1}$ coefficientwise. If $\|v_k\|_{\mathcal{A}} \le 1$, then submultiplicativity of the $\|\cdot\|_{\mathcal{A}}$ norm gives

$$\|v_{k+1}\|_{\mathcal{A}} \le \frac{\Gamma+\delta}{a} \le 1, \qquad \forall\, k \ge 0.$$

Coefficientwise monotone convergence therefore gives a $k \to \infty$ limit $v \in \mathcal{A}$ with $\|v\|_{\mathcal{A}} \le 1$: monotone convergence gives $\sum_{k\ge1} [t^k] v(t) \le 1$, and coefficientwise passage in the recursion gives $v = (\Gamma t + r^{\#}(v))/a$. The formal equation

$$\eta = \frac{\Gamma t - r(\eta)}{a} \tag{3.13}$$

has a unique formal power series solution with zero constant term, since r has order at least three. Induction on degree gives $|[t^k]\eta| \le [t^k]v$ for every k, so $\eta \in \mathcal{A}$ and $\|\eta\|_{\mathcal{A}} \le 1$. Absolute convergence upgrades the formal identity (3.13) to $q(\eta(t)) = \Gamma t$ in $\mathcal{A}$.

By Lemma 3.5, η^{phys} is allowable. Radial extension respects composition: writing $\eta(t) = tA(t^2)$ and $q(t) = tB(t^2)$, one has

$$q^{\text{phys}}\big(\eta^{\text{phys}}(z)\big) = zA(|z|^2)B\big(|z|^2 A(|z|^2)^2\big) = (q \circ \eta)^{\text{phys}}(z) = \Gamma z.$$

Apply the valid kernel represented by q to the two families of feature vectors realizing η^{phys}. This proves validity of the composed linear kernel. $\square$

4. Choice of Kernels

4.1. **Preprocessing the phase kernel.** We now describe the first five branches of our mixed rounding scheme, which use phase rounding after different preprocessings. Let

$$\rho(t) = \sum_{m\ge0} c_m t^{2m+1} = t\, p(t^2), \qquad p(x) = \sum_{m\ge0} c_m x^m, \qquad c_m \in \mathbb{R}, \quad \sum_{m\ge0} |c_m| \le 1.$$

By the existence lemma for valid kernels (Lemma 3.5), the radial extension $\rho^{\text{phys}}(z) = z\, p(|z|^2)$ is an allowable preprocessor. Composing this preprocessor with phase rounding produces the valid kernel

$$K_\rho(z) = \mathfrak{h}\big(\rho^{\text{phys}}(z)\big).$$

Its odd analytic representative is $h \circ \rho$. For example, Haagerup's preprocessor $\rho_*(t) = h^{-1}(\gamma_H t)$ belongs to this family and gives the exactly linear kernel $K_{\rho_*}(z) = \gamma_H z$. To compute the coefficients of K_ρ, put $x = |z|^2$ and $\alpha_j = h_j/\pi$. Since p has real coefficients, the expansion of $\mathfrak{h}$ gives

$$K_\rho(z) = \pi z \sum_{j\geq 0} \alpha_j x^j p(x)^{2j+1}, \qquad \alpha_0 = \frac{1}{4}, \qquad \frac{\alpha_{j+1}}{\alpha_j} = \frac{(2j+1)^2}{4(j+1)(j+2)}. \tag{4.1}$$

In particular, when the coefficients of ρ are rational, the coefficients of K_ρ can be computed exactly over $\mathbb{Q}$, apart from the common factor π.

4.2. **The sixth kernel.** We now define the sixth alternative. Let $P(s) = \sum_{j=0}^{9} P_j s^j$, where the following displayed decimals are exact terminating rationals:

$$\begin{aligned}(\Re P_0, \dots, \Re P_9) = (&-0.08698387653870097, -1.2330310998031548, 0.7731241827583651,\\ &-0.22194697498795207, 0.02812176806963062, -0.000900511816912206,\\ &-0.00016426639538512312, 0.000019346729645617798, -0.00000078064685429180 22,\\ &0.000000010807705028012223),\\ (\Im P_0, \dots, \Im P_9) = (&-0.9438672688639868, 1.7787014426224697, -1.2883230878434588,\\ &0.37871541533769754, -0.046163071426838245, 0.00035393667390378955,\\ &0.0004808219961183191, -0.000048050193159213 06, 0.0000018538964354898714,\\ &-0.000000025226107769319 2).\end{aligned}$$

A simple univariate polynomial certificate proves

$$|P(s)| > \frac{3}{5} \qquad \forall\, s \geq 0. \tag{4.2}$$

Put $u(s) := P(s)/|P(s)|$. For any $t \in \mathbb{R}$, write $\mathrm{cay}(t) = \frac{1-t^2+2it}{1+t^2} \in S(\mathbb{C})$. Define the following exact parameters:

$$C := \mathrm{cay}\left(\frac{356080252073}{10^{12}}\right), \qquad A := \mathrm{cay}\left(-\frac{1775}{10000}\right), \qquad \delta := \frac{11}{500}, \tag{4.3}$$

$$u_0 := u(0), \qquad z_0 := \overline{C}\, u_0^2, \qquad D := -iz_0, \qquad B := DA. \tag{4.4}$$

Although u_0 need not have rational coordinates, u_0^2 does; hence so do z_0, D, B. Define

$$f(s) := u(s)\,\mathrm{phase}(s + \delta A), \qquad g(s) := C\overline{u(s)}\,\mathrm{phase}(s + \delta B), \tag{4.5}$$

$$F(z) := \mathrm{phase}(z) f(|z|^2), \qquad G(z) := \mathrm{phase}(z) g(|z|^2). \tag{4.6}$$

Note that $F, G \colon \mathbb{C} \to S(\mathbb{C})$. Equations (4.2) and $\Im A, \Im B \neq 0$ show that these phases are well-defined away from irrelevant Gaussian null sets. Moreover,

$$f(0)\overline{g(0)} = z_0 A \overline{B} = i. \tag{4.7}$$

This exact cancellation removes the slow h_n term from the real-coefficient tail obtained by the conjugation symmetrization below.

By Lemma 3.5, their direct phase-rounding kernel is valid. It is charge-one equivariant, since $K(z) = \mathbb{E}[F(X_z)\overline{G(Y_z)}]$, so for any $\omega \in S(\mathbb{C})$, since $(\omega X_z, Y_z)$ has covariance ωz, we have $K_{F,G}(\omega z) = \mathbb{E}[F(\omega X_z)\overline{G(Y_z)}] = \omega K_{F,G}(z)$.

Lemma 4.1. *There exist complex numbers $(c_k)_{k\geq 0}$ with $\sum_{k\geq 0} |c_k| \leq 1$ such that*

$$K(z) = K_{F,G}(z) = \sum_{k\geq 0} c_k z^{k+1} \overline{z}^k, \qquad \forall\, z \in \overline{\mathbb{D}}. \tag{4.8}$$

Proof. Let $L_k^{(1)}$ be the generalized Laguerre polynomial and define

$$L_k^{(1)}(x) := \frac{x^{-1}e^x}{k!}\frac{d^k}{dx^k}(e^{-x}x^{k+1}), \qquad \phi_k(z) := \frac{zL_k^{(1)}(|z|^2)}{\sqrt{k+1}}. \tag{4.9}$$

$$\begin{aligned} a_k &:= \int_0^\infty e^{-s}\frac{\sqrt{s}L_k^{(1)}(s)}{\sqrt{k+1}}f(s)\,ds = \langle F,\phi_k\rangle, \\ b_k &:= \int_0^\infty e^{-s}\frac{\sqrt{s}L_k^{(1)}(s)}{\sqrt{k+1}}g(s)\,ds = \langle G,\phi_k\rangle. \end{aligned} \tag{4.10}$$

Then $(\phi_k)_{k\geq 0}$ are normalized charge-one complex Hermite functions, since

$$\int_0^\infty se^{-s}L_j^{(1)}(s)L_k^{(1)}(s)\,ds = (k+1)\mathbf{1}_{\{k=j\}}.$$

In particular, $\mathbb{E}|\phi_k(Z)|^2 = 1$. By completeness of the generalized Laguerre polynomials, $(\phi_k)_{k\geq 0}$ is an orthonormal basis of the charge-one subspace of complex Gaussian L_2.

Also, $\phi_k(\omega z) = \omega\phi_k(z)$ for all $\omega \in S(\mathbb{C})$. By Laguerre orthogonality, ϕ_k is orthogonal to every charge-one polynomial of smaller degree. Also ϕ_k has bidegree $(k+1,k)$, so the span of ϕ_k is the charge-one Hermite component $\mathcal{H}_{k+1,k}$. Let Π_k denote the orthogonal projection onto the span of ϕ_k. Here all function inner products and L_2 norms are taken with respect to standard complex Gaussian measure.

Let $z \in \overline{\mathbb{D}}$. The Mehler identity says if $H \in \mathcal{H}_{p,q}$ is a Hermite component with bidegree p,q, then $T_zH = z^p\overline{z}^qH$, where

$$T_z\psi(y) := \mathbb{E}\psi(zy + W\sqrt{1-|z|^2}), \qquad \forall\, y\in\mathbb{C},\ \forall\,\psi\colon \mathbb{C}\to\mathbb{C} \text{ with } \mathbb{E}|\psi(Y)|^2<\infty,$$

where Y, W are i.i.d. standard complex Gaussians, $X_z := zY + W\sqrt{1-|z|^2}$ and $\mathbb{E}X_z\overline{Y} = z$,

$$K(z) \overset{(3.7)}{=} \mathbb{E}[F(X_z)\overline{G(Y)}] = \langle T_zF, G\rangle. \tag{4.11}$$

Since F, G are charge-one equivariant almost surely ($F(\omega z) = \omega F(z)$ for all $\omega \in S(\mathbb{C})$ and a.e. $z\in\mathbb{C}$), their Hermite decompositions have only Π_k components, i.e. $F = \sum_{k\geq 0}\Pi_kF$, hence $T_zF = \sum_{k\geq 0} z^{k+1}\overline{z}^k\Pi_kF$, so that

$$K(z) \overset{(4.11)}{=} \langle T_zF, G\rangle = \sum_{k\geq 0} z^{k+1}\overline{z}^k\langle \Pi_kF, G\rangle = \sum_{k\geq 0} z^{k+1}\overline{z}^k\langle\Pi_kF,\Pi_kG\rangle. \tag{4.12}$$

Define now for any $k\geq 0$,

$$c_k := \langle \Pi_kF, \Pi_kG\rangle \overset{(4.9)}{=} \langle F,\phi_k\rangle\langle\phi_k, G\rangle \overset{(4.10)}{=} a_k\overline{b_k}. \tag{4.13}$$

By (4.13), Hermite orthogonality and the Cauchy–Schwarz inequality give

$$\sum_{k\geq 0}|c_k| \leq \Big(\sum_{k\geq 0}\|\Pi_kF\|_2^2\Big)^{1/2}\Big(\sum_{j\geq 0}\|\Pi_jG\|_2^2\Big)^{1/2} \leq 1. \tag{4.14}$$

∎

Let

$$F^{\#}(z) := \overline{F(\overline{z})}, \qquad G^{\#}(z) := \overline{G(\overline{z})}, \qquad \forall\, z\in\mathbb{C}.$$

In contrast to (4.8), $K_{F^{\#},G^{\#}}(z) = \sum_{k\geq 0}\overline{c_k}z^{k+1}\overline{z}^{\,k}$ by (4.13) since $\phi_k(\overline{z}) = \overline{\phi_k(z)}$, $\forall\ k\geq 0$.

By Lemma 3.5,

$$K_{\mathrm{org}} := (K_{F,G} + K_{F\#,G\#})/2 \tag{4.15}$$

is a valid kernel. Moreover, its n^{th} coefficient is

$$k_n := (c_n + \overline{c_n})/2 = \Re(c_n) \in \mathbb{R}. \tag{4.16}$$

Finally, the triangle inequality implies that $\sum_{n\geq 0} |k_n| \leq 1$.

5. The mixed rounding scheme

In this section and the two certification sections, ρ_r, $1 \leq r \leq 5$, denotes an odd analytic representative in $\mathcal{A}$, and ρ_r^{phys} denotes its radial extension. For any $\rho \in \mathcal{A}$ whose radial extension is allowable, write

$$K_\rho(z) := \mathfrak{h}\big(\rho^{\mathrm{phys}}(z)\big). \tag{5.1}$$

The five scalar preprocessors used in the final candidate are now defined directly. Every decimal in (5.2) is an exact terminating rational, not a rounded display value:

$$\begin{aligned}
\rho_1(t) &= 0.906231648312t - 0.093651693922t^3 - 0.000116657762t^9,\\
\rho_2(t) &= 0.910377315558t - 0.063651630829t^3 - 0.025971053610t^7,\\
\rho_3(t) &= 0.912512978184t - 0.035570362029t^3 - 0.050646289677t^7 - 0.001270370107t^9,\\
\rho_4(t) &= 0.901913867354t - 0.037111896010t^3 - 0.026391249065t^7\\
&\quad - 0.033420629518t^{11} - 0.001162358049t^{15}.
\end{aligned} \tag{5.2}$$

Note that these are different perturbations of ρ_* from (2.7), since by [Haa87],

$$\rho_*(t) \approx 0.9062789294t - 0.093045561673t^3 - 0.00049038053603t^7 - 0.00012586558875t^9 - \cdots$$

For the fifth (high-degree) preprocessor, put

$$\rho_5(t) = \sum_m c_{5,m} t^{2m+1}, \tag{5.3}$$

where every coefficient not listed here is zero and the listed values are again exact:

m	$c_{5,m}$	m	$c_{5,m}$	m	$c_{5,m}$
0	0.094626045381	1	−0.856254413568	10	0.001274576105
13	0.008312996734	14	0.001885453067	15	0.001662844635
16	0.009295485863	17	0.009399425108	18	0.008540450681
19	0.000847985940	20	0.003771690693	21	0.000873880329
27	−0.000419180236	29	−0.000311954497	30	−0.000455650279
31	−0.000996284704	35	−0.000865568525	37	0.000206113648

(5.4)

Thus, if $\rho_r(t) = \sum_m c_{r,m} t^{2m+1}$, the actual correlation applied to inner products is

$$\rho_r^{\mathrm{phys}}(z) = \sum_m c_{r,m} z|z|^{2m}, \qquad r \in \{1,2,3,4,5\}. \tag{5.5}$$

Exact summation gives the especially simple Wiener norms

$$\|\rho_1\|_{\mathcal{A}} = \|\rho_4\|_{\mathcal{A}} = \frac{999999999996}{10^{12}}, \|\rho_2\|_{\mathcal{A}} = \|\rho_3\|_{\mathcal{A}} = \frac{999999999997}{10^{12}}, \|\rho_5\|_{\mathcal{A}} = \frac{999999999993}{10^{12}}. \tag{5.6}$$

Consequently all preprocessors $\{\rho_r^{\mathrm{phys}}\}_{r=1}^5$ are allowable by Lemma 3.5, and each K_{ρ_r} is a valid Gaussian phase-rounding kernel for each $1 \leq r \leq 5$.

With the sixth kernel from (4.15), label the corresponding top-level kernels by

$$K_r := K_{\rho_r} \quad (1 \le r \le 5), \qquad K_6 := K_{\mathrm{org}}, \qquad K_{6,+} := K_{F,G}, \quad K_{6,-} := K_{F\#,G\#}. \tag{5.7}$$

Thus $K_6 = (K_{6,+} + K_{6,-})/2$ by (4.15). Define also

$$\begin{aligned} w_1 &:= 0.983543086263686583, & w_2 &:= 0.0124174018143118538, \\ w_3 &:= 0.0030311909556774751, & w_4 &:= 0.000593632758261406976, \\ w_5 &:= 0.0000169358532612748746, & w_6 &:= 0.000397752354801503847. \end{aligned} \tag{5.8}$$

The unnormalized kernel used in Theorem 1.1 is then

$$\widehat{Q} := \sum_{r=1}^{6} w_r K_r = \sum_{r=1}^{5} w_r K_r \quad + \frac{w_6}{2} K_{6,+} + \frac{w_6}{2} K_{6,-}. \tag{5.9}$$

6. Certifying the Upper Bound

6.1. **Bounds for the first 120 coefficients.** Let

$$\widehat{Q}(z) = \sum_{k \ge 0} \widehat{q}_k z |z|^{2k} \tag{6.1}$$

be the sum defined in (5.9). By Lemmas 3.6 and 3.2,

$$K_G^{\mathbb{C}} \le \frac{\sum_{r=1}^{6} w_r}{\widehat{q}_0 - \sum_{k \ge 1} |\widehat{q}_k|} \qquad \text{when the denominator is positive.}$$

The following estimates bound this denominator from below.

The rational part of every scalar contribution to $\widehat{q}_0, \dots, \widehat{q}_{119}$ is computed exactly from (4.1), and the common factor π is enclosed by directed ball arithmetic. The contribution of K_{org} to the sum (6.1) is estimated by bounding the coefficients (4.10). Here and below, hats mark coefficients of the raw, unnormalized candidate. Combining these bounds with the weights in (5.8) gives

$$\sum_{k=1}^{119} |\widehat{q}_k| < 1.074829404498064 \cdot 10^{-6}, \qquad \widehat{q}_0 > 0.71179640805640433452. \tag{6.2}$$

The bounds on (4.10) integrate over $0 \le s \le 25$, using the substitution $s = t^2$. A second table bounds the integrals over $25 \le s \le 36$; the verifier adds the corresponding interval enclosures before forming products. If $\widetilde{a}_n, \widetilde{b}_n$ are the two coefficient sequences whose integrals are truncated at $s = 36$, and $e_n^{(a)} = a_n - \widetilde{a}_n$, $e_n^{(b)} = b_n - \widetilde{b}_n$ are the omitted sequences, Bessel's inequality together with $|F| = |G| = 1$ give

$$\|(e_n^{(a)})_{n \ge 0}\|_{\ell_2} \le \|F 1_{|z|^2 > 36}\|_{L_2} \le e^{-18}, \qquad \|(e_n^{(b)})_{n \ge 0}\|_{\ell_2} \le \|G 1_{|z|^2 > 36}\|_{L_2} \le e^{-18}. \tag{6.3}$$

Thus, for any finite index block $\mathcal{I}$,

$$\sum_{n \in \mathcal{I}} |a_n \overline{b_n} - \widetilde{a}_n \overline{\widetilde{b}_n}| \le e^{-18} \big(\|\widetilde{a}_{\mathcal{I}}\|_2 + \|\widetilde{b}_{\mathcal{I}}\|_2 \big) + e^{-36}. \tag{6.4}$$

The file `complex_upper_origin_layer_coefficients_n120_analytic.npz` is the accepted interval table. The companion table is `origin_tail_25_36.npz`. Writing $P = R + iI$, where R and I are the real-coefficient polynomials $\Re P$ and $\Im P$, the table generator evaluates them separately and uses the analytic continuation $(R + iI)/\sqrt{R^2 + I^2}$ on complex integration balls. This point is important: projecting the real and imaginary parts of a single complex ball inside an analytic integration callback would not be a valid holomorphic continuation.

6.2. **Tail bound for the sixth kernel.** We now bound $\sum_{n\geq 120}|k_n|$, where k_n is defined in (4.16). The Laguerre integral

$$\int_0^\infty e^{-s}\frac{\sqrt{s}L_n^{(1)}(s)}{\sqrt{n+1}}\,ds \overset{(4.10)\wedge(4.12)\wedge(2.1)}{=} \sqrt{h_n}, \qquad \forall\, n\geq 0$$

shows that the coefficient of the constant radial amplitude phase(z) is $\xi_n := \sqrt{h_n}$, consistently with (4.10). Write

$$\begin{aligned} a_n &= f(0)\xi_n + r_n, \qquad b_n = g(0)\xi_n + t_n,\\ R_f(z) &:= \mathrm{phase}(z)[f(|z|^2) - f(0)] =: \sum_{n\geq 0} r_n\phi_n,\\ R_g(z) &:= \mathrm{phase}(z)[g(|z|^2) - g(0)] =: \sum_{n\geq 0} t_n\phi_n. \end{aligned} \tag{6.5}$$

Recall from (4.13) that $c_n = a_n\overline{b_n}$ and $k_n = \Re(c_n)$ from (4.16). Substituting (6.5) and using (4.7), together with $\xi_n = \sqrt{h_n}$, gives

$$k_n = \Re\Big[f(0)\overline{g(0)}\,h_n + f(0)\xi_n\overline{t_n} + r_n\overline{g(0)}\,\xi_n + r_n\overline{t_n}\Big] = \Re\Big[f(0)\xi_n\overline{t_n} + r_n\overline{g(0)}\,\xi_n + r_n\overline{t_n}\Big].$$

Since $|f(0)| = |g(0)| = 1$ and $|\xi_n| = \sqrt{h_n}$, it follows that

$$|k_n| \leq \sqrt{h_n}\big(|r_n| + |t_n|\big) + |r_n||t_n|. \tag{6.6}$$

Denote $z\in\mathbb{C}$ by $z = x + y\sqrt{-1}$, $x, y\in\mathbb{R}$ and define $\mathcal{N} := -\frac{1}{2}(\partial_x^2 + \partial_y^2) + x\partial_x + y\partial_y$. Write $z = re^{i\theta}$ and $s = r^2$. In polar coordinates, $\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$. If $\Phi(z) = e^{i\theta}v(s)$, then $\Delta\Phi = e^{i\theta}(4sv''(s) + 4v'(s) - v(s)/s)$ and $(x\partial_x + y\partial_y)\Phi = r\partial_r\Phi = e^{i\theta}2sv'(s)$. Consequently,

$$\mathcal{N}\big(v(s)\,\mathrm{phase}(z)\big) = \mathrm{phase}(z)\left[-2sv''(s) + (2s-2)v'(s) + \frac{v(s)}{2s}\right]. \tag{6.7}$$

When $v(s) = \sqrt{s}L_n^{(1)}(s)$, the Laguerre differential equation $sL_n^{(1)\prime\prime}(s) + (2-s)L_n^{(1)\prime}(s) + nL_n^{(1)}(s) = 0$ with (6.7) says

$$-2sv''(s) + (2s-2)v'(s) + \frac{v(s)}{2s} = \sqrt{s}\Big(L_n^{(1)}(s) + 2(s-2)L_n^{(1)\prime}(s) - 2sL_n^{(1)\prime\prime}(s)\Big) = (2n+1)v(s).$$

Consequently, by (4.9),

$$\mathcal{N}\phi_n = (2n+1)\phi_n, \qquad \forall\, n\geq 0. \tag{6.8}$$

We apply (6.7) to $v = f - f(0)$ and $v = g - g(0)$, where f, g are defined in (4.5). The nonvanishing estimate (4.2) and $\Im A, \Im B \neq 0$ by (4.3), (4.4) imply that f and g are twice continuously differentiable on $[0,\infty)$, so

$$f(s) - f(0) = O(s), \qquad g(s) - g(0) = O(s) \qquad (s\downarrow 0).$$

For our choice of v, the square brackets in (6.7) are then bounded near $s = 0$ and square integrable against $e^{-s}\,ds$. Since the phase function is discontinuous at zero, we can use a smooth cutoff near zero (and infinity, since v and its derivatives have polynomial growth) which then converges in the norm $\sqrt{\|\cdot\|_2^2+\|\mathcal{N}(\cdot)\|_2^2}$ by (6.7). Hence $R_f, R_g \in \mathrm{Dom}(\mathcal{N})$. Then (6.8) implies that

$$\|\mathcal{N}R_f\|_2^2 \overset{(6.5)\wedge(6.8)}{=} \sum_{n\geq 0}(2n+1)^2|r_n|^2, \qquad \|\mathcal{N}R_g\|_2^2 = \sum_{n\geq 0}(2n+1)^2|t_n|^2. \tag{6.9}$$

Directed integration, including explicit near-zero and far-tail estimates, proves

$$\|\mathcal{N}R_f\|_2^2 < 12.671, \qquad \|\mathcal{N}R_g\|_2^2 < 12.674. \tag{6.10}$$

$$\sum_{n<120}(2n+1)^2|r_n|^2 > 11.497, \qquad \sum_{n<120}(2n+1)^2|t_n|^2 > 11.498. \tag{6.11}$$

Thus the two tail energies are less than 1.174 and 1.176. Lemma 6.1 and (2.2) imply

$$W_{120} := \sum_{n\geq 120}\frac{h_n}{(2n+1)^2} \leq \frac{1}{48\cdot 119^3}. \tag{6.12}$$

Applying Cauchy–Schwarz to (6.6) while multiplying and dividing by $(2n+1)$ gives

$$T_6 := \sum_{n\geq 120}|k_n| \leq \sqrt{1.174\,W_{120}} + \sqrt{1.176\,W_{120}} + \frac{\sqrt{1.174\cdot 1.176}}{241^2} < \frac{523}{2{,}000{,}000} = 2.615\cdot 10^{-4}. \tag{6.13}$$

Lemma 6.1 (Wallis tails)**.** *Recall h_k from (2.2). For every integer $N \geq 1$,*

$$\sum_{k\geq N} h_k \leq \frac{1}{4N}. \tag{6.14}$$

Moreover, for all $N \geq 2$,

$$\sum_{k\geq N}\frac{h_k}{(2k+1)^2} \leq \frac{1}{48(N-1)^3}. \tag{6.15}$$

Proof. Wallis' central-binomial estimate $\binom{2k}{k}^2/16^k \leq 1/(\pi k)$, together with (2.2), gives

$$h_k \leq \frac{1}{4k(k+1)} \qquad \forall\, k \geq 1.$$

Using $1/(k(k+1)) = 1/k - 1/(k+1)$, $\sum_{k\geq N} 1/(k(k+1)) = 1/N$, proving (6.14). Also,

$$\frac{h_k}{(2k+1)^2} \leq \frac{1}{16k^3(k+1)} \leq \frac{1}{16k^4}.$$

Comparison with $\int_{N-1}^{\infty} x^{-4}\,dx$ proves (6.15). □

6.3. **Tail bound for the fifth (high-degree) preprocessor.** As in (4.8), write

$$K_5(z) \overset{(3.6)}{=} \mathfrak{h}(\rho_5^{\mathrm{phys}}(z)) \overset{(5.7)}{=:} \sum_{n\geq 0} k_{5,n} z|z|^{2n}. \tag{6.16}$$

In this section we will show that

$$\tau_5 := w_5 \sum_{n\geq 120}|k_{5,n}| < 6.319618149454563\cdot 10^{-8}. \tag{6.17}$$

Write $\rho_5^{\mathrm{phys}}(z) = zp_5(|z|^2)$, as specified in (5.3) and (5.4) whose largest nonzero coefficient index is 37. The polynomial $x^j p_5(x)^{2j+1}$ in (4.1) has degree at most

$$j + (2j+1)37 = 75j + 37. \tag{6.18}$$

Thus the blocks $0 \le j < 67$ have a tail contribution bounded by indices $120 \le n \le 5000$ in (6.17). From (2.3), we have $h(\rho_5(t)) = \sum_{j\ge 0} h_j(\rho_5(t))^{2j+1}$. Combined with (6.16), we have

$$k_{5,n} = [t^{2n+1}] \sum_{j\ge 0} h_j \rho_5(t)^{2j+1}.$$

Directed evaluation gives

$$\sum_{n=120}^{5000} \left| w_5 [t^{2n+1}] \sum_{j=0}^{66} h_j \rho_5(t)^{2j+1} \right| < 2.699176355797438 \cdot 10^{-12}. \tag{6.19}$$

Recall $\|\rho_5\|_{\mathcal{A}} \le 1$, so (5.6), (3.3) and Lemma 6.1 bound tail terms omitted from (6.19) by

$$w_5 \sum_{j\ge 67} |h_j| \|\rho_5^{2j+1}\|_{\mathcal{A}} \le w_5 \sum_{j\ge 67} |h_j| \le \frac{|w_5|}{4\cdot 67} < 6.3193482318189831 \cdot 10^{-8}. \tag{6.20}$$

Combining (6.19) and (6.20) gives (6.17).

6.4. **Tail bound for the first four preprocessors.** In this section, we bound the first four terms in (5.9). For any $1 \le r \le 4$, write $\rho_r^{\mathrm{phys}}(z) = zp_r(|z|^2)$ and $\rho_r^{\mathrm{an}}(\zeta) := \rho_r(\zeta) = \zeta p_r(\zeta^2)$. Use the radii

$$R_1 = \frac{6}{5}, \qquad R_2 = R_3 = R_4 = \frac{111}{100}. \tag{6.21}$$

By (2.4), $h(w)$ is equal to the following expression for all $w \in (-1,1)$

$$h_\Omega(w) := w \int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{1 - w^2 \sin^2\theta}} d\theta. \tag{6.22}$$

Denote $\Omega := \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$. Then h_Ω is well-defined and holomorphic for any $w \in \Omega$, since for all $0 \le v \le 1$, $1 - vw^2 \notin (-\infty, 0]$ (if $1 - vw^2 \in (-\infty,0]$, then $vw^2 \in [1,\infty)$, violating $w \in \Omega$). In defining h_Ω, we set $\sqrt{1} := 1$. Meanwhile, we claim that

$$\rho_r^{\mathrm{an}}(\overline{R_r\mathbb{D}}) \subset \Omega. \tag{6.23}$$

Given this claim, we can input $w := \rho_r^{\mathrm{an}}(\zeta)$ into h_Ω in (6.22) for any $\zeta \in \overline{R_r\mathbb{D}}$. To prove (6.23), the code verifies that, for all $1 \le r \le 4$, for any $|\zeta| = R_r$ with $w := \rho_r^{\mathrm{an}}(\zeta)$,

$$|w| = |\rho_r^{\mathrm{an}}(\zeta)| \overset{(5.5)}{\le} \sum_m |c_{r,m}| R_r^{2m+1} < 5/4, \quad \text{and also} \quad \min_{0\le v\le 1} |1 - vw^2| > 1/100. \tag{6.24}$$

For any $0 \le v \le 1$, define $F_{r,v}(\zeta) := 1 - v \cdot (\rho_r^{\mathrm{an}}(\zeta))^2$. By (6.24), we may define

$$N_r(v) := \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta| = R_r} \frac{F'_{r,v}(\zeta)}{F_{r,v}(\zeta)} \, d\zeta,$$

which is the number of zeros of $F_{r,v}$ in the disc $|\zeta| \le R_r$ (no zeros occur on the boundary $|\zeta| = R_r$ by (6.24)). From (6.24), N_r is continuous in v and integer-valued, but $F_{r,0} = 1$, so $N_r(v) = 0$ for all $0 \le v \le 1$, hence $F_{r,v}(\zeta)$ has no zeros in $|\zeta| \le R_r$ for all $0 \le v \le 1$. So, if $\rho_r^{\mathrm{an}}(\zeta) = x \in \mathbb{R}$ with $|x| \ge 1$, then $v := x^{-2} \in (0,1]$ implies $F_{r,v}(\zeta) = 1 - x^2 x^{-2} = 0$, a contradiction. That is, we proved (6.23). Thus (6.22) holds for $w = \rho_r^{\mathrm{an}}(\zeta)$ and $|\zeta| \le R_r$, so

$$|h_\Omega(w)| \le |w| \int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{|1-w^2\sin^2\theta|}} d\theta < \frac{5}{4} \cdot 10 \int_0^{\pi/2} \cos^2\theta d\theta = 25\pi/8 < 10.$$

Recall $K_r(z) \overset{(3.6)}{=} \mathfrak{h}(\rho_r^{\text{phys}}(z)) =: \sum_{n\ge 0} k_{r,n} z|z|^{2n}$. Define $H_r(\zeta) := h_\Omega(\rho_r^{\text{an}}(\zeta))$. Then $H_r(\zeta) = \sum_{n\ge 0} k_{r,n}\zeta^{2n+1}$, and we showed that $|H_r(\zeta)| < 10$ for all $|\zeta| = R_r$.

Cauchy's estimate then implies that, for all $1 \le r \le 4$,

$$T_r := \sum_{n\ge 120} |k_{r,n}| \le 10 \sum_{n\ge 120} R_r^{-(2n+1)} = \frac{10R_r^{-241}}{1-R_r^{-2}}. \tag{6.25}$$

Their total raw weighted contribution is

$$\sum_{r=1}^{4} |w_r| T_r < 1.017179644647 \cdot 10^{-11}. \tag{6.26}$$

Finally, (6.13) and the origin weight give

$$|w_6| T_6 < 1.040122407805933 \cdot 10^{-7}. \tag{6.27}$$

6.5. **Putting together the upper bound.**

Proof of Theorem 1.1, Upper Bound. The sum of the unnormalized weights is

$$\mathcal{M} := \sum_{r=1}^{6} w_r \overset{(5.8)}{=} \frac{125000000000000121997}{125000000000000000000} = 1.000000000000000975976. \tag{6.28}$$

The normalized selector probabilities are $w_r/\mathcal{M}$. Exact comparison gives

$$\mathcal{M} < 1.0000000000000001. \tag{6.29}$$

The combined tail bound for $K_1, \dots, K_4, K_6$ is

$$T_{\text{fix}} := \sum_{r=1}^{4} |w_r| T_r + |w_6| \frac{523}{2{,}000{,}000} \overset{(6.26)\wedge(6.27)}{<} 1.040224125770398 \cdot 10^{-7}. \tag{6.30}$$

Let $\widehat{E} = (0, \widehat{q}_1, \widehat{q}_2, \dots)$ be the nonlinear coefficient vector of $\widehat{Q}$. The prefix, τ_5, and fixed-tail estimates give

$$\|\widehat{E}\|_1 \overset{(6.2)\wedge(6.17)\wedge(6.30)}{<} 1.242047998569650 \cdot 10^{-6}. \tag{6.31}$$

Since $\widehat{Q}/\mathcal{M}$ is a convex combination of valid kernels by (5.9), (5.6) and (4.15), Lemma 3.5 shows that it is a valid kernel. Its analytic representative has the form

$$q(t) = \frac{\widehat{q}_0}{\mathcal{M}} t + r(t), \qquad \|r\|_{\mathcal{A}} = \frac{\|\widehat{E}\|_1}{\mathcal{M}}. \tag{6.32}$$

Since $\|\widehat{E}\|_1 < \widehat{q}_0$ by (6.2) and (6.31), Lemma 3.6 applies. It supplies an allowable common preprocessor η^{phys} and an exactly linear valid kernel $K(z) = \Gamma z$, where one may take

$$\Gamma := \frac{\widehat{q}_0 - \|\widehat{E}\|_1}{\mathcal{M}}. \tag{6.33}$$

Then the deliberately rounded bounds (6.2), (6.29), and (6.31) give

$$\Gamma > \frac{0.7117964080564 - 1.24204800 \cdot 10^{-6}}{1.0000000000000001} > 0.7117951660083 > \frac{1}{1.404898555}. \tag{6.34}$$

Then displayed bounds (6.2) and (6.31) together with (6.28) give the sharper bound

$$\Gamma > 0.71179516600840569540. \tag{6.35}$$

Lemma 3.2 now gives $K_G^{\mathbb{C}} \leq \Gamma^{-1} < 1.404898555$. In fact the code verifies that

$$\Gamma^{-1} < 1.404898554745441. \tag{6.36}$$

This proves the upper bound of Theorem 1.1 since [Haa87] showed $K_G^{\mathbb{C}} \leq \gamma_H^{-1}$, $\gamma_H < 0.71178980668499814812$, so $\gamma_H^{-1} > 1.404909132735795$, and $\gamma_H^{-1} - \Gamma^{-1} > 1.057799 \cdot 10^{-5}$. □

7. The Lower Bound

In this section, function norms and expectations use standard complex Gaussian measure, with density $\pi^{-n} e^{-\|z\|^2}$ on $\mathbb{C}^n$. We write γ_1 for this measure when $n = 1$.

Recall that the Hermite space $\mathcal{H}_{p,q}$ consists of the Wick-ordered polynomials of holomorphic degree p and antiholomorphic degree q; these spaces are mutually orthogonal. Write $P_{p,q}$ for their orthogonal projections and set $P = P_{1,0}$, $Q = P_{2,1}$. Throughout

$$\varepsilon = \frac{37}{50}, \qquad \rho = \frac{49}{50}, \qquad T = T_n = P - \varepsilon Q.$$

On vector-valued functions, T_n acts on each coordinate. The lower bound of Theorem 1.1 follows from the following.

Theorem 7.1.

$$\sup_{n \geq 1} \|T\|_{L_\infty(\mathbb{C}^n) \to L_1(\mathbb{C}^n)} < \frac{2857}{4000} < \frac{5}{7}.$$

Proof of lower bound of Theorem 1.1 assuming Theorem 7.1. Here Z is a standard complex Gaussian vector in $\mathbb{C}^n$. The first inequality below follows from Grothendieck's inequality (see e.g. [Hei26b, JM26]).

$$K_G^{\mathbb{C}} \geq \sup_{n \geq 1} \frac{\sup_{g\colon \mathbb{C}^n \to S(\mathbb{C}^n)} \mathbb{E}\|Tg(Z)\|_{\ell_2(\mathbb{C}^n)}}{\sup_{f\colon \mathbb{C}^n \to S(\mathbb{C})} \mathbb{E}|Tf(Z)|} \geq \frac{1}{\sup_{n \geq 1} \|T\|_{L_\infty(\mathbb{C}^n) \to L_1(\mathbb{C}^n)}}. \tag{7.1}$$

For the second inequality, take $g(z) = z/\|z\|$ for $z \neq 0$, with any unit value at zero. Then

$$\left\| g - \frac{z}{\sqrt{n}} \right\|_2^2 = \mathbb{E}\left(1 - \frac{\|Z\|}{\sqrt{n}}\right)^2 \leq \mathbb{E}\left(1 - \frac{\|Z\|^2}{n}\right)^2 = \frac{1}{n}.$$

Since $T_n(z/\sqrt{n}) = z/\sqrt{n}$ and $\|T_n\|_{2\to 2} = 1$, $\|T_n g - g\|_2 \leq 2/\sqrt{n}$. Thus $\mathbb{E}\|T_n g(Z)\| \to 1$ as $n \to \infty$, proving the second inequality. Consequently $K_G^{\mathbb{C}} > 4000/2857 > 7/5$ follows by Theorem 7.1. □

We now proceed to prove Theorem 7.1. Let $f, g\colon \mathbb{C}^n \to S(\mathbb{C})$. We will upper bound $\Re\langle f, T_n g\rangle$. Since T_n is selfadjoint with charge-one range, successive replacements $f = \mathrm{phase}(T_n g)$ and $g = \mathrm{phase}(T_n f)$ do not decrease $\mathrm{Re}\langle f, T_n g\rangle$, so we may assume that f, g are charge-one. (If either polynomial whose phase is taken is identically zero, then $\Re\langle f, T_n g\rangle \leq 0$ and Theorem 7.1 is immediate. Otherwise its zero set has Gaussian measure zero, so its phase is charge-one a.e.) Put $h := (f+g)/2$, $k := (f-g)/2$. Denote $t := \|Ph\|_2$. Then

$$\begin{gathered} |h|^2 + |k|^2 = 1, \qquad \Re(\bar{h}k) = 0, \\ \Re\langle f, T_n g\rangle = t^2 - \varepsilon\|Qh\|_2^2 + \varepsilon\|Qk\|_2^2 - \|Pk\|_2^2. \end{gathered} \tag{7.2}$$

Since $|h| \le 1$, we have $0 \le t \le \sup_{\|r\|_2 \le 1} |\langle Ph, r\rangle| = \sup_{\|r\|_2\le 1} |\langle h, Pr\rangle| \le \mathbb{E}|W| = \sqrt{\pi}/2 < 9/10$, where $W \in \mathbb{C}$ is a standard complex Gaussian.

Section 8.3 proves that, for every $q \in \mathcal{H}_{2,1}$ with $\mathbb{E}|q|^2 = 1$ and $m := \mathbb{E}|q|^4$,

$$2 \le m \le 62, \qquad \mathbb{E}|q|^6 \le 9m - 12 + 50(m-2)^{3/2}, \qquad \mathbb{E}|q| \le \rho. \tag{7.3}$$

In particular $\|Qu\|_2 \le \rho$ for any $u\colon \mathbb{C}^n \to \mathbb{C}$ with $|u| \le 1$. Discarding the negative terms in (7.2) gives $\Re\langle f, T_n g\rangle \le t^2 + \varepsilon\rho^2 = t^2 + .710696$.

8. Certifying the Lower Bound

8.1. **Reduction to a scalar potential.** Write a point of $\mathbb{C}^n$ as $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$. (If $n = 1$, w can be a single point.) Recall $t := \|Ph\|_2$. By applying a unitary transformation, we may assume $Ph(z, w) = tz$. Denote

$$U := |z|^2, \qquad A := U - 1, \qquad B := \frac{z^2}{\sqrt{2}}, \qquad H := \frac{z(U-2)}{\sqrt{2}}.$$

The functions $1, A, B, z, H, \bar{z}$ are orthonormal in $L^2(\gamma_1; \mathbb{C})$. Recalling $k = (f - g)/2$, set

$$c_D(w) := \mathbb{E}_z[\overline{D(z)}k(z, w)],\ \forall\, w \in \mathbb{C}^{n-1} \qquad \xi := (c_1, c_A, \overline{c_B})^T, \quad \zeta := (c_z, c_H)^T, \quad d := c_{\bar{z}}.$$

A *profile* consists of $0 < \delta < \varepsilon$, real ℓ, β, and real symmetric positive definite matrices $E \in \mathbb{R}^{3\times 3}$ and $R \in \mathbb{R}^{2\times 2}$ satisfying

$$E \succeq \mathrm{diag}(-1, \varepsilon, \varepsilon), \quad E \succeq \mathrm{diag}(\varepsilon - \delta, 0, 0), \quad R \succeq \mathrm{diag}(-1, \varepsilon), \quad R \succeq \mathrm{diag}(\varepsilon, 0). \tag{8.1}$$

Write

$$\mathcal{E}(k) := \xi^* E \xi + \zeta^* R \zeta + \varepsilon |c_{\bar{z}}|^2.$$

Denote Q_w as the operator Q acting only on the w coordinates. Hermite orthogonality gives

$$\varepsilon \|Qk\|_2^2 - \|Pk\|_2^2 \le \mathbb{E}_w \mathcal{E}(k) + \delta \|Q_w c_1\|_2^2. \tag{8.2}$$

To prove this, denote $\Pi_{p,q}$ as the w-variable Hermite projection, and write

$$Pk = \Pi_{1,0}c_1 + z\Pi_{0,0}c_z, \qquad Qk = \Pi_{2,1}c_1 + A\Pi_{1,0}c_A + B\Pi_{0,1}c_B + z\Pi_{1,1}c_z + H\Pi_{0,0}c_H + \bar{z}\Pi_{2,0}c_{\bar{z}}$$

In the expansion of Pk, the term involving z is $z\Pi_{0,0}c_z$, and the remaining terms are $\Pi_{1,0}c_1$. Denote $\xi_{p,q} := \Pi_{p,q}\xi$. Then $\mathbb{E}_w \xi^* E \xi = \sum_{p,q} \mathbb{E}_w \xi_{p,q}^* E \xi_{p,q}$. Apply the first part of (8.1) to $\xi_{1,0}$ and the second to $\xi_{2,1}$ (using also $\Pi_{1,0}\overline{c_B} = \overline{\Pi_{0,1}c_B}$) to get

$$\mathbb{E}_w \xi^* E \xi \ge -\|\Pi_{1,0}c_1\|_2^2 + \varepsilon \|\Pi_{1,0}c_A\|_2^2 + \varepsilon\|\Pi_{0,1}c_B\|_2^2 + (\varepsilon - \delta)\|\Pi_{2,1}c_1\|_2^2. \tag{8.3}$$

Applying a similar argument to R then gives

$$\mathbb{E}_w \zeta^* R \zeta \ge -|\Pi_{0,0}c_z|^2 + \varepsilon |\Pi_{0,0}c_H|^2 + \varepsilon \|\Pi_{1,1}c_z\|_2^2. \tag{8.4}$$

Adding (8.3) and (8.4) to $\varepsilon \mathbb{E}_w |c_{\bar{z}}|^2 \ge \varepsilon \|\Pi_{2,0}c_{\bar{z}}\|_2^2$ and adding $\delta\|Q_w c_1\|_2^2$ proves (8.2).

Put $L(z) := z(\ell + \beta(U - 2))$. Since $\mathbb{E}\bar{z}h = t$ and $\|z(U-2)\|_2^2 = 2$, we have

$$0 \le \varepsilon \|Qh + \beta\sqrt{2}H/\varepsilon\|_2^2 = \varepsilon\|Qh\|_2^2 + 2\Re\mathbb{E}[\overline{\beta z(U-2)}Qh] + 2\beta^2\|H\|_2^2/\varepsilon.$$

That is,

$$-\varepsilon \|Qh\|_2^2 \le 2\Re\,\mathbb{E}[\overline{\beta z(U-2)}h] + \frac{2\beta^2}{\varepsilon}. \tag{8.5}$$

Define the particular residual function

$$q := \delta Q_w c_1, \qquad \sigma := \|q\|_2 \le \delta\rho, \qquad \delta\|Q_w c_1\|_2^2 = 2\Re\mathbb{E}_w \bar{q} c_1 - \sigma^2/\delta. \tag{8.6}$$

The bound on σ follows from $|c_1| \le 1$ and (7.3). The equality follows since q is in the range of Q_w and $c_1 = Q_w c_1 + (I - Q_w)c_1$, so $\mathbb{E}_w \bar{q} c_1 = \mathbb{E}_w \bar{q} Q_w c_1 = \delta \|Q_w c_1\|_2^2 \in \mathbb{R}$ and $\sigma^2/\delta = \delta\|Q_w c_1\|_2^2$.

For a complex number $\omega \in \mathbb{C}$, define the scalar potential

$$\Psi(\omega) := \frac{2\beta^2}{\varepsilon} + \sup_{f_0, g_0 \colon \mathbb{C} \to S(\mathbb{C})} \Big(\mathcal{E}(k_0) + 2\Re\mathbb{E}_z[\bar{L}h_0] + 2\Re\mathbb{E}_z[\bar{\omega}k_0(z)]\Big). \tag{8.7}$$

Here f_0, g_0 depend only on z, with $h_0 := (f_0+g_0)/2$, $k_0 := (f_0 - g_0)/2$. Bessel's inequality makes the supremum finite. We claim that $\Psi(\omega) = \Psi(|\omega|)$, which follows since the map

$$(f_0(z), g_0(z)) \mapsto e^{-i\theta}(f_0(e^{i\theta}z), g_0(e^{i\theta}z)),$$

multiplies ξ by $e^{-i\theta}$, ζ is fixed, and d is multiplied by $e^{-2i\theta}$. Hence the $\mathcal{E}$ and L terms in (8.7) stay fixed while the coefficient of $\overline{\omega}$ can be rotated by any phase. Substituting (8.5) and (8.6) into (7.2), then using (8.2) and this observation with $\omega := q(w)$,

$$\Re\langle f, T_n g\rangle \le t^2 - 2\ell t + \mathbb{E}_w \Psi(|q|) - \frac{\sigma^2}{\delta}. \tag{8.8}$$

The potential $s \mapsto \Psi(s)$, $s \in \mathbb{R}$, is convex and 2-Lipschitz: it is a supremum of affine functions with slopes $2\Re c_1 \in [-2, 2]$. It is even by the claim, hence nondecreasing on $[0, \infty)$.

8.2. **Finite optimization.** We optimize (8.8). Set $K := (\operatorname{Re}\xi, \operatorname{Im}\xi, \operatorname{Re}\zeta, \operatorname{Im}\zeta, \operatorname{Re}d, \operatorname{Im}d)^T \in \mathbb{R}^{12}$. Then $\mathcal{E}(k_0) = K^T M K$ and $K_j = \Re\mathbb{E}_z \overline{\mathcal{B}_j} k_0$ for all $1 \le j \le 12$ where

$$M := \operatorname{diag}(E, E, R, R, \varepsilon I_2) \in \mathbb{R}^{12\times 12}, \qquad \mathcal{B} := (1, A, B,\ i, iA, -iB,\ z, H,\ iz, iH,\ \bar{z}, i\bar{z}).$$

Let $v \in \mathbb{R}^{12}$. Note that $2v^T K - v^T M^{-1} v = K^T M K - (v - MK)^T M^{-1}(v - MK) \le K^T M K$ with equality when $v = MK$, so $K^T M K = \sup_{v\in\mathbb{R}^{12}}(2v^T K - v^T M^{-1} v)$. Adding $2v^T K$ to the last two terms in (8.7) yields $2\Re\mathbb{E}_z[\overline{\omega + \mathcal{B}v}k_0 + \bar{L}h_0] = \Re\mathbb{E}_z[\overline{\omega + \mathcal{B}v + L}f_0 + \overline{-\omega - \mathcal{B}v + L}g_0]$. For any $w > 0$, we have $0 \le (|a| - w)^2 = |a|^2 - 2|a|w + w^2$, so that $|a| \le (w + |a|^2/w)/2$. Applying this for positive step functions w_+, w_-, gives in (8.7), for any $s \in \mathbb{R}$

$$\Psi(s) \le \sup_{v\in\mathbb{R}^{12}} R_s(v), \qquad R_s(v) := \frac{2\beta^2}{\varepsilon} + \frac{1}{2}\mathbb{E}_z \sum_{\eta\in\{-1,1\}} \left(w_\eta + \frac{|s + \mathcal{B}v + \eta L|^2}{w_\eta}\right) - v^T M^{-1} v. \tag{8.9}$$

Then $R_s(v)$ is a quadratic polynomial in v. At each rational node s_j, the code expands it as $c_j + 2b_j^T v - v^T D_j v$ and certifies $D_j \succ 0$, $c_j + b_j^T D_j^{-1} b_j \le A_j$. Then $\Psi(s_j) \le A_j$ since $R_{s_j}(v) = c_j + b_j^T D_j^{-1} b_j - (v - D_j^{-1}b_j)^T D_j (v - D_j^{-1} b_j) \le c_j + b_j^T D_j^{-1} b_j$. Convexity of Ψ bounds Ψ below the line segment between consecutive nodes $0 = s_0 < \cdots < s_J = S$; the 2-Lipschitz bound gives $\Psi(s) \le A_J + 2(s - S)$ for $s \ge S$.

The code supplies polynomials $p(s) = a_0 + a_1 s + a_2 s^2 + a_4 s^4 + a_6 s^6$, with $a_1, a_6 \ge 0$, lying above these line segments and the entire affine tail. Thus $p \ge \Psi$. For $\sigma > 0$ put $x^2 := \mathbb{E}|q/\sigma|^4 - 2$. By (7.3),

$$\begin{aligned}\mathbb{E}\Psi(|q|) - \sigma^2/\delta \le J_p(\sigma, x) &:= a_0 + \rho a_1 \sigma + (a_2 - \delta^{-1})\sigma^2 \\ &\quad + a_4(2 + x^2)\sigma^4 + a_6(6 + 9x^2 + 50x^3)\sigma^6.\end{aligned} \tag{8.10}$$

Here $0 \le \sigma \le \delta\rho$, $0 \le x \le \sqrt{60} < 31/4$. If $\sigma = 0$, then $q = 0$ and $\mathbb{E}\Psi(|q|) \le p(0)$. The coefficients a_2, a_4 may have either sign because those moments are exact. On each rectangle of a complete cover of $[0, \delta\rho] \times [0, 31/4]$, one such polynomial is checked to satisfy $J_p \le C$.

For fixed σ, the x-dependence is $ax^2 + bx^3$ with $b \geq 0$, so its maximum is at an endpoint by the first derivative test. Consequently (8.8) becomes

$$\Re\langle f, T_n g\rangle \leq t^2 - 2\ell t + C. \tag{8.11}$$

The table gives the nine certified pairs (ℓ_i, C_i) and, at $i = 0$, the elementary pair $(0, \varepsilon\rho^2)$. All decimals are exact rationals; the remaining profile parameters and full certificates are supplied with the verifier code.

i	ℓ_i	C_i	i	ℓ_i	C_i
0	0	0.710696	5	0.610085330	1.080649
1	0.121848821	0.725165	6	0.697084166	1.197222
2	0.232185104	0.760598	7	0.782190305	1.325111
3	0.333426881	0.816862	8	0.833308193	1.408127
4	0.492185281	0.946281	9	0.884543194	1.495777

Set $\tau_0 := 0$, $\tau_i := \frac{C_i - C_{i-1}}{2(\ell_i - \ell_{i-1})}$, $\forall\, 1 \leq i \leq 9$, $\tau_{10} := 9/10$. The checker verifies that these endpoints increase and that $F_i(t) = t^2 - 2\ell_i t + C_i$ is strictly below $2857/4000$ at both endpoints of $[\tau_i, \tau_{i+1}]$. Convexity proves the same bound on each interval, completing Theorem 7.1.

8.3. **The cubic moment lemma.**

Proof of (7.3). We first prove $\kappa_6 \leq 68\kappa_4^{3/2}$ in (8.13). We then take $X = \sqrt{2}\Re q$ for $q \in \mathcal{H}_{2,1}$ with $\|q\|_2 = 1$, derive the moment bounds in (7.3), and conclude with $\mathbb{E}|q| \leq 49/50$.

A real symmetric tensor f of order 3 on $\mathbb{R}^n$ is an array $f = (f_{abc})_{1\leq a,b,c\leq n}$ whose entries are invariant under permutation, e.g. $f_{abc} = f_{acb} = f_{bca} = f_{bac} = f_{cba} = f_{cab}$. Let $Y_1, \ldots, Y_n$ be independent real standard Gaussian random variables. The Wick cubic $I_3(f)$ is $I_3(f) = \sum_{1\leq a,b,c\leq n} f_{abc}(Y_aY_bY_c - 1_{\{a=b\}}Y_c - 1_{\{a=c\}}Y_b - 1_{\{c=b\}}Y_a)$. Denote $\|f\|^2 := \sum_{1\leq a,b,c\leq n} f_{abc}^2$. For a real symmetric tensor f, denote $X := I_3(f)$. Define

$$F_{ab,c} := f_{abc}, \quad G := F^TF, \quad S := \mathrm{sym}_4(FF^T), \quad s := \|G\|^2,$$
$$h := \|S\|^2, \quad D := FG, \quad U := SF,$$

where sym_4 averages all permutations of the four tensor indices and norms are Hilbert–Schmidt, and F is treated as an $n^2 \times n$ matrix. Also put

$$V := \|D\|^2 = \mathrm{tr}(G^3), \qquad B_0 := \|F^T\,\mathrm{vec}(G)\|^2, \qquad A_0 := \sum_{a,b,c,d,e=1}^{n} f_{abc}f_{dec}G_{ad}G_{be}.$$

Write $\kappa_4 := \mathbb{E}X^4 - 3(\mathbb{E}X^2)^2$ and $\kappa_6 := \mathbb{E}X^6 - 15\mathbb{E}X^4\mathbb{E}X^2 + 30(\mathbb{E}X^2)^3$. We check

$$\begin{aligned} \kappa_4 &= 1296(s + 3h/2), \\ \frac{\kappa_6}{1296^{3/2}} &= \frac{135}{2}\,\mathrm{tr}(S^3) + 25V + \frac{45}{2}B_0 + \frac{165}{2}\langle D, U\rangle + 45\|U\|^2 + 45(A_0 - V). \end{aligned} \tag{8.12}$$

Wick's rule and cumulant cancellation express κ_r as the sum over connected loopless cubic multigraphs on r labelled vertices. An edge is a summed tensor index; a graph with multiplicities e_{ij} has weight $(3!)^r/\prod_{i<j} e_{ij}!$. The program enumerates all such graphs for $r = 4, 6$, expands the right-hand sides of (8.12) into the same contractions, and compares exact rational coefficients after relabelling vertices. Symmetry of f makes this a formal identity, with no dimension bound or numerical tensor tests.

Since sym_4 averages over norm-preserving index permutations, $h = \|\mathrm{sym}_4(FF^T)\|^2 \le \|FF^T\|^2 = s$, so $0 \le h \le s$. If G has eigenvalues (λ_i), then $s = \sum \lambda_i^2$, so $\|F\|_{\mathrm{op}}^2 = \|G\|_{\mathrm{op}} = \max \lambda_i \le \sqrt{s}$. Cauchy–Schwarz and the eigenvalue bounds give

$$A_0 \le V \le s^{3/2}, \quad B_0 \le s^{3/2}, \quad \mathrm{tr}(S^3) \le h^{3/2}, \quad \|U\|^2 \le h\sqrt{s}, \quad \langle D, U\rangle \le s\sqrt{h}.$$

For the first inequality, diagonalize G: then $A_0 = \sum_{ijk} \lambda_i\lambda_j f_{ijk}^2 \le \sum_{ijk} \lambda_k^2 f_{ijk}^2 = V$ by $2\lambda_i\lambda_j \le \lambda_i^2 + \lambda_j^2$ and symmetry. Then $V = \sum \lambda_i^3 \le \max \lambda_i \cdot s \le s^{3/2}$. Next, $B_0 \le \|F\|_{\mathrm{op}}^2 \|G\|^2 \le s \cdot \sqrt{s}$. The $\mathrm{tr}(S^3)$ bound follows similarly by diagonalizing S. The $\|U\|^2$ bound mimics the B_0 bound. The final bound follows by Cauchy–Schwarz. Thus, with $y := \sqrt{h/s} \in [0,1]$,

$$\frac{\kappa_6}{\kappa_4^{3/2}} = \frac{\kappa_6/s^{3/2}}{\kappa_4^{3/2}/s^{3/2}} \overset{(8.12)}{\le} \frac{N(y)}{(1+3y^2/2)^{3/2}} < 68, \qquad N(y) := \frac{135}{2}y^3 + \frac{95}{2} + \frac{165}{2}y + 45y^2. \quad (8.13)$$

The last step is a rational polynomial check: $68^2(1+3y^2/2)^3 - N(y)^2 > 0$ on $[0,1]$. If $s = 0$, then $f = 0$ and the cumulant inequality is immediate.

For any $q \in \mathcal{H}_{2,1}$ with $\|q\|_2 = 1$, circular symmetry makes $X = \sqrt{2}\Re q = (q+\bar{q})/\sqrt{2}$ a normalized real third-chaos variable, with $\mathbb{E}X^2 = 1$, $\mathbb{E}X^4 = (3/2)m$, $\mathbb{E}X^6 = (5/2)\mathbb{E}|q|^6$, so

$$\kappa_4 = \tfrac{3}{2}(m-2), \qquad \kappa_6 = \tfrac{5}{2}(\mathbb{E}|q|^6 - 9m + 12), \qquad m := \mathbb{E}|q|^4. \quad (8.14)$$

Since $\kappa_4 \ge 0$ by (8.12), it follows that $m \ge 2$. Also $\mathbb{E}|q|^6 \le 9m - 12 + 50(m-2)^{3/2}$ by (8.13) and (8.14), using $68(3\sqrt{6}/10) < 50$. Also $h \le s \le (\mathrm{tr}\, G)^2 = \|f\|^4 = 1/36$ since $1 = \mathbb{E}X^2 = 3!\|f\|^2$. So $\kappa_4 \le 90$ by (8.12) and therefore $m \le 62$ by (8.14).

Finally, put $Y := |q|$, $\mu = \mathbb{E}Y$ and $n_6 := \mathbb{E}Y^6$. Cauchy–Schwarz, the triangle inequality, $\mathbb{E}Y^4(Y+1)^2 = \mathbb{E}Y^6 + 2\mathbb{E}Y^5 + \mathbb{E}Y^4$, $\mathbb{E}Y^5 \le \sqrt{\mathbb{E}Y^6\mathbb{E}Y^4}$ and $2\sqrt{n_6 m} \le n_6/6 + 6m$ give

$$\begin{aligned}(m-1)^2 &= \big(\mathbb{E}(Y-1)[Y^2(Y+1)]\big)^2 \le \mathbb{E}(Y-1)^2 \cdot \mathbb{E}Y^4(Y+1)^2 \\ &\le 2(1-\mu)(\sqrt{n_6} + \sqrt{m})^2 \le 2(1-\mu)(7n_6/6 + 7m).\end{aligned} \quad (8.15)$$

For $x = \sqrt{m-2}$, the $\mathbb{E}|q|^6$ bound of (7.3) gives $n_6 \le 6 + 9x^2 + 50x^3$, so

$$\begin{aligned}25(m-1)^2 - 7n_6/6 - 7m &\ge 25x^4 - \tfrac{175}{3}x^3 + \tfrac{65}{2}x^2 + 4 \\ &= 25(x^2 - 7x/6 - 1/12)^2 + (95/36)(x - 35/38)^2 + 181/114 > 0.\end{aligned} \quad (8.16)$$

Combining (8.15) and (8.16) yields $1 - \mu \ge \frac{(m-1)^2}{2((7/6)n_6 + 7m)} \ge 1/50$, so $\mu \le 49/50$. For $q = Qu$ and $|u| \le 1$, $\|q\|_2^2 = \Re\mathbb{E}\bar{q}u \le \mathbb{E}|q| \le (49/50)\|q\|_2$, completing (7.3).

□

9. Supporting Numerical Codes

The numerical portion of the upper bound of Theorem 1.1 can be run with the command: `python3 audit_best7_full.py` with the codes available at: https://github.com/sheilman77/grothendieck_cx. The numerical portion of the lower bound for Theorem 1.1 can be run with the command: `python3 check_complex_lower.py`.

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