HEARING THE SHAPE OF A DRUM: RECENT RESULTS

STEVEN M. HEILMAN

1. Introduction

Were the world inscribed in nature's language, could we but decipher what is heard? Such is the quest of mathematics: to find truth. Below, I hope to describe an advance in our understanding of that which is true.

We begin with the question posed in 1966 by esteemed mathematician Mark Kac [K]: "Can one hear the shape of a drum?" A sensible response is, "Why should we care?" As Kac shows in his article, the equation that describes a vibrating object (the so-called wave equation) is of fundamental importance[1]. Studied for hundreds of years, it reappears in many fields and guises in mathematics, physics, etc. and is of fundamental importance. You can investigate the wave equation yourself by, say, stretching out a Slinky and shaking it or playing a musical instrument. Moreover, investigation of the wave equation closely informs that of the (also fundamentally important) heat equation[2] which relates to Brownian motion, etc., and, when properly adjusted, gives us Schrödinger's equation of quantum mechanics. One can think of the heat and wave equations as prototypes, playing a similar role as maize and fruit flies in genetics.

From the mathematical perspective, Kac really asks," How well do we understand the wave equation?" Since it took almost thirty years to answer Kac's question (in the negative, see Fig. 1), it seems that we do not have a complete understanding of this (deceptively simple) equation.

Steven M. Heilman is a graduate student of mathematics at the Courant Institute of Mathematical Sciences, New York University. He was supported by the National Science Foundation through the Research Experiences for Undergraduates Program at Cornell. His email address is `heilman@cims.nyu.edu`.

[1]The mathematical set-up for the wave is equations is the following: let n a positive integer, $T > 0$, $u = u(x,t)\colon D \times (0,T) \to \mathbb{R}$, $D \subset \mathbb{R}^n$, D compact with smooth boundary. Solve $\sum_{j=1}^{n} \frac{\partial^2}{\partial {x_i}^2} u =: \Delta u = \frac{\partial^2 u}{\partial t}$, where we choose one of two boundary conditions: $u|_{\partial D} = 0$ (Dirichlet boundary conditions) or $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$ (Neumann boundary conditions), where ν denotes the derivative of u in the normal direction.

[2]To describe this equation mathematically, take the same set-up as above, but solve the equation $\Delta u = \frac{\partial u}{\partial t}$. In both cases, the investigation of solutions of $(-\Delta)u = \lambda u$ are of closely related.

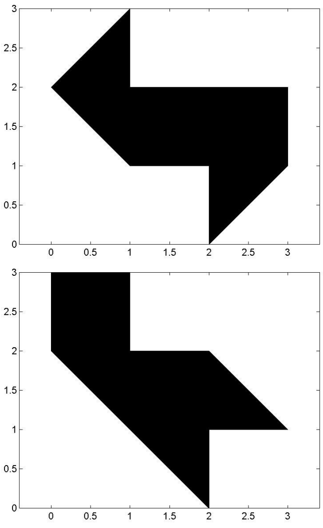

Figure 1. Two drums that sound the same

In 1992, Gordon, Webb and Wolpert helped our understanding of this equation, by discovering two drums which vibrate in exactly the same way. However, it would be nice to have some positive evidence, of the following form: if a drum satisfies certain hypotheses, then I can reconstruct the drum just by listening to it. This is the kind of result that Steve Zelditch has found. We wish to describe it below, after a brief history of previous results.

2. Statement of Problem

Suppose we have a drum that has a flat surface and has no holes. For simplicity, we ignore any effects of the drum cavity itself, i.e. we examine only the flat, vibrating surface. A musician's intuition (and/or mathematician's proof) says that the drum vibrates at an infinite number of frequencies (some of which may occur repeatedly due to "independent" vibrations of the same frequency). In musical language, these frequencies are the harmonics: the fundamental together with its overtones. Label these frequencies in increasing order $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. From this infinite increasing sequence of numbers, we wish to recover as much information as we can about the drum surface. Put another way:

Problem 1: We want to find if two different domains are in fact different, just by looking at their frequencies of vibration.

Extending the linguistic analogy from the introduction, we don't want to find any words (i.e. drums) that are homophones.

This problem seems hard if not impossible to solve directly, so let us see if we can say anything at all about it. Using Weyl's law, we know that the rate of growth of the frequencies depends on the surface area and dimension of the drum. With this law, our ear can (theoretically) differentiate a drum from a string, or a drum of surface area 2 from another drum with surface area 2.3. Even fancier mathematical technology shows that we can hear the perimeter of the drum and certain quantities involving the curvature of the drum's boundary[3].

In Fig. 1, we see a pair of drum surfaces with the same vibrational frequencies (so-called *isospectral* domains) [GWW]. One can check that the surface areas and perimeters of the drums are the same. One also sees that the edges of each are flat. Since these surfaces look very odd, one may wonder how they were found. Well, the crucial part of the argument comes from an excellent paper of Sunada from 1985. Here, a certain abstract condition is given for finding isospectral drums. The proof of the formula follows by imitating an analogous argument in number theory, oddly enough. With this result, Gordon et al. found some explicit examples of 2D isospectral objects in 3D space. Finding these examples required some clever geometry and symmetry considerations. They then flattened these examples, and came up with, among other things, Fig. 1.

Since the paper [GWW], many other examples of drums with identical vibrational modes have been found. Moreover, the proofs have been greatly simplified, so that only symmetry and planar geometry are manipulated. In particular, to prove that the two drums have the same vibrational frequencies, you do the following: (i) cut one drum into pieces (as in Fig. 2), (ii) piece it back together in an appropriate way, and (iii) that's it! The results here (and below) are interesting, because they show the (typical?) behavior of mathematical solutions to difficult problems. We begin with a question in one area, and may have to go through seemingly unrelated fields (number theory in the above case) to arrive at our solution.

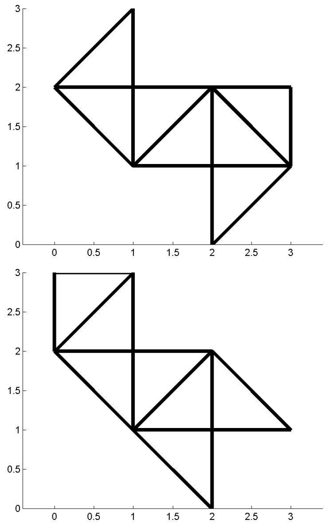

Figure 2. Two domains that sound the same, decomposed

3. Positive Evidence: When Math and Physics Collide

With the negative evidence above in mind, we can now describe Zelditch's approach to the positive kind. The plan is as described in Section 1. We want to find a class of drum surfaces so that, given the vibrational frequencies of an unknown drum in the class, we can determine which drum we are hearing. From the counterexamples in Section 2, we see that this class cannot include all drum surfaces. We also know that all counterexamples have sharp corners and lack symmetry. With these considerations in mind, we are led to the following:

Theorem 1 (Informal statement, due to Zelditch, 2009) Consider the class of drum surfaces with no holes and very smooth[4] boundary and with at least one mirror symmetry. Then, given the vibrational frequencies of an unknown drum, we *can* reconstruct the drum.

Finally, after all those negative results (which are still useful in guiding our understanding) we get something positive! Now, how do we prove something like this? After some though, it seems we cannot reconstruct our drum from a finite number of vibrational frequencies. We really need to use an infinite number of them. In order to do this, we take a sort of infinite sum of the frequencies. Consider the following analogy. In number theory, people are very interested in the integers and the primes. To study all the integers (and primes) at once, number theorists examine the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

[3]Here I am referring to the coefficients of the asymptotic expansion about $t = 0$ of the heat trace (or trace of the heat kernel) $Z(t) = Tre^{t\Delta} = \sum_{j=0}^{\infty} e^{-\lambda_j t}$). This relates, for instance, to the partition function of statistical mechanics.

[4]To be technically correct, we need the boundary to be analytic. There are other technical assumptions as well, one of which we discuss below, but this statement gives the general idea.

which is initially defined for $s > 1$, so that the sum converges (as we recall from calculus). It turns out that studying the behavior of $\zeta(s)$ as we let s decrease to 1, gives us information about the growth rates of primes[5]. Doing similar things for an infinite set of frequencies tell us a lot about (say) the growth rates of the λ_i.

The general strategy follows thusly. (i) Lump all of our frequencies $(\lambda_1, \lambda_2, \ldots)$ together in some amalgamated function (as in the definition of the zeta function with the integers). (ii) Study the behavior of this function at points that address all $(\lambda_1, \lambda_2, \ldots)$ at once (for instance, do something like a Taylor series expansion). (iii) See what comes out of the calculations.

There is a final ingredient in this theorem that I have so far not discussed. Take the drum surface and put vertical walls around the boundary. Now, picture a small billiard ball rolling around inside and reflecting off of the edges, without friction. (You can find frictionless balls in the absolute zero freezer next to the massless springs.) It turns out that the long-time behavior of the ball's trajectory has immense implications for the behavior of the vibrational frequencies. After some thought, this may make sense because the trajectory of the ball is affected by properties of the boundary of our drum, and so are the vibrations. There are deeper reasons to expect this relation which involve quantum mechanics, but we do not have time to discuss these issues here.

We are now ready to summarize Zelditch's argument [Z].

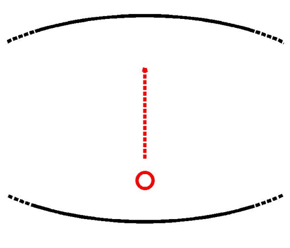

FIGURE 3. A bouncing ball, as in Step 1

Step 1. By the symmetry assumption, there is a line which is fixed by the mirror symmetry. If a ball rolls along this line, it will bounce back and forth forever, in a repetitive manner (see Fig. 3).
Step 2. Isolate the behavior of this bouncing ball with an amalgamated function, as in the definition of the zeta function $\zeta(s)$. (To do this, we need to make an additional assumption: the length of one period of this bouncing ball's path is unique in a certain sense.)
Step 3. Do an analogue of a Taylor series[6] for this amalgamated function. (This is the most difficult step, by far[7]).
Step 4. From these "Taylor series" coefficients, a (surprisingly) simple inductive argument allows us to recover the Taylor coefficients *of the boundary of the drum* where the bouncing ball strikes the boundary.
Step 5. By the very smooth boundary assumption, we can recover the boundary entirely from the Taylor coefficients of the drum at a single point. QED. (Note that the entire proof follows by properties of our amalgamated function. This function depends only on the frequencies λ_i of the drum.)

4. CONCLUSION: FUTURE DIRECTIONS

The argument as stated above applies equally well to higher dimensional domains, with appropriate adjustments [HZ]. However, the story is not complete. For instance, the reader may object, since we essentially assume what we need, given the present technology, to complete the proof. This may seem like cheating, but the selection of the proper assumptions may require a lot of work. Also, one could say that the identification of required assumptions constitutes an understanding of the problem. For the future, one can try to weaken these assumptions and still maintain the result. Apparently such work is in progress. And as we hear more clearly nature's tongue, we learn to speak in ways thus far unknown.

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[5] And we get even more information by letting $s \in \mathbb{C}$ and examining the behavior along the line $\Re(s) = 1$, as in the complex analytic proof of the prime number theorem.

[6] In fact, we need to do an asymptotic expansion.

[7] Mathematically, we begin with a type of resolvent of the Laplacian, which is frequency localized around the bouncing ball orbit. Then, several reductions are needed to explicitly calculate the asymptotic expansion of this resolvent. We reduce from the resolvent to a boundary integral operator. From there, we then reduce to asymptotic estimates for an oscillatory integral.