

RECENT WORK ON THE PROPELLER CONJECTURE

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ABSTRACT. How can one prove a sharp inequality? Symmetrization, Fourier analysis, and probability are often used, and we will survey some of these methods through examples. We then survey sharp constants in Grothendieck inequalities, leading to some recent work on computing the best constant for a Grothendieck-type inequality of Khot and Naor. (Joint work with Aukosh Jagannath and Assaf Naor)

1. INTRODUCTION: HOW TO PROVE A SHARP INEQUALITY

The first sharp inequality that we would like to discuss is the planar isoperimetric inequality. We present Hurwitz's proof. The first step is to express a parameterized curve as a sum of simpler pieces (complex exponentials). Then, in the course of the proof, we see that the area enclosed by the curve is maximized if and only if our curve is a sum of the lowest order complex exponentials. Luckily, the latter quantity is still a curve. In other problems that we consider, the class of objects that we are trying to optimize may not have this property. That is, if we try to apply this same paradigm, the sum of lowest order terms may exit the class of objects under consideration.

Theorem 1.1. (*Hurwitz's Proof of the Isoperimetric Inequality, 1901*) *Let γ be a closed, continuous, non self-intersecting and rectifiable curve, $\gamma: [0, 1] \rightarrow \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$ with length $\ell = 1$ and unit speed. That is $\sup_{0=t_0 < t_1 < \dots < t_n=1} \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) = 1$ and $(x')^2 + (y')^2 = 1$. Let A denote the area of the interior of Γ . Then $A \leq \frac{1}{4\pi}$, with equality iff γ is a circle.*

Proof. (Summary) $\gamma(t)$ is of bounded variation, and is therefore a.e. differentiable. Write $x(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t}$, $y(t) = \sum_{n \in \mathbb{Z}} b_n e^{2\pi i n t}$. Then $x'(t) \sim \sum_{n \in \mathbb{Z}} 2\pi i n a_n e^{2\pi i n t}$, $y'(t) \sim \sum_{n \in \mathbb{Z}} 2\pi i n b_n e^{2\pi i n t}$.

$$\begin{aligned} \text{Area}(\text{Int}(\gamma)) &= \int_{\text{Int}(\gamma)} dx dy = \frac{1}{2} \left| \int_0^1 (x(t)y'(t) - y(t)x'(t)) dt \right|, \text{ Stokes} \\ &= \frac{1}{2} \left| 2\pi i \sum_{n \in \mathbb{Z}} n (a_n \bar{b}_n + b_n \bar{a}_n) \right|, \text{ Parseval} \\ &\leq \pi \sum_{n \in \mathbb{Z}} |n| (|a_n|^2 + |b_n|^2), \text{ AMGM} \\ &\leq \pi \sum_{n \in \mathbb{Z}} |n|^2 (|a_n|^2 + |b_n|^2) = \frac{1}{4\pi} \int_0^1 ((x'(t))^2 + (y'(t))^2) dt = \frac{1}{4\pi}, \end{aligned}$$

by Parseval. For the equality case, note that $a_n = b_n = 0$ for $|n| \geq 2$ □

Date: September 28, 2011.

Claim 1.2. An inequality is not fully understood unless it is sharp.

It already becomes apparent from the definition of the Theorem that there are two parts to such a result. First, one has to deal with several technicalities. Then, one tries to optimize the quantity that you are dealing with.

It would be very nice if the above proof worked in higher dimensions. But so far, no one has figured out how to accomplish this. The problem seems to be similar to the one I described above. Given a high dimensional region A , you can express 1_A in terms of its Hermite coefficients. But if you try to optimize a function of these coefficients, you may end up with something that is no longer the indicator function of a set.

Problem 1.3. Generalize the above proof to higher dimensions.

Another approach [L] to this (vaguely stated) problem comes from rewriting the isoperimetric inequality. For $f \in L_1(\mathbb{R}^n)$, define

$$T_t f(x) := \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy.$$

Then the isoperimetric inequality in \mathbb{R}^n (if $|B| = |A|$ and B is a ball, then $|\partial B| \leq |\partial A|$) is implied by the following statement

$$\|T_t(1_A)\|_2 \leq \|T_t(1_B)\|_2, \quad \forall t \geq 0. \tag{1.1}$$

On the real side, since T_t essentially takes averages over balls, this statement is basically a continuous version of Brunn-Minkowski: $|\lambda A + (1 - \lambda)B|^{1/n} \geq \lambda |A|^{1/n} + (1 - \lambda) |B|^{1/n}$. (Perimeter is measured by $\langle T_t 1_{A^c}, 1_A \rangle = -\langle T_t 1_A, 1_A \rangle + \langle T_t 1, 1_A \rangle = -\langle T_t 1_A, 1_A \rangle + |A|$. This is why the directions of the inequality signs are reversed, for (1.1) and Brunn-Minkowski.) However, (1.1) becomes much more interesting when we take the Fourier transform ($\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$) and apply Parseval:

$$\|\widehat{1_A}(\xi) \cdot e^{-4t\pi^2 \xi^2}\|_2 \leq \|\widehat{1_B}(\xi) \cdot e^{-4t\pi^2 \xi^2}\|_2, \quad \forall t \geq 0.$$

So, among all indicator functions, the Fourier transform of the ball is “most concentrated” around the origin. This statement is somehow a converse to the decay $|\widehat{1_B}(\xi)| \leq c_n |\xi|^{(1-n)/2}$ given by Stationary Phase (Theorem 5.3).

Problem 1.4. Prove (1.1) directly.

2. MORE EXAMPLES OF SHARP INEQUALITIES

In the past few weeks, we have seen several examples and methods about sharp inequalities. Since we have seen a lot about symmetrization, I am going to de-emphasize these results in this talk.

We now discuss Ball’s cube slicing result. Let $H \subset \mathbb{R}^n$ be a hyperplane through the origin. We want a sharp upper bound for the $|H \cap [-1/2, 1/2]^n|$. The first step is to define a one variable function. Let $\xi = (\xi_1, \dots, \xi_n) \in S^{n-1}$ be a unit normal to H . For $t \in \mathbb{R}$ define

$$A(t) := \left| (H + t\xi) \cap \left[-\frac{1}{2}, \frac{1}{2} \right]^n \right|.$$

Let $g = 1_{[-1/2, 1/2]}$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, by definition of $\|x\|_\infty$,

$$1_{[-1/2, 1/2]^n}(x) = g(\|x\|_\infty) = \prod_{i=1}^n g(|x_i|). \quad (*)$$

As we will see, Fourier inversion for A turns the problem into a (tractable) integral inequality. This result is surprising since \widehat{A} is a certain average of A , and it is somewhat counterintuitive that an average of all values of A says something useful about a single value of A .

Theorem 2.1. (*Ball's Cube Slicing, 1986*) $A(t) \leq \sqrt{2}$, with equality if and only if H contains an $(n-2)$ -dimensional face of $[-1/2, 1/2]^n$, and H contains the origin.

Proof. (Sketch) We first calculate $\widehat{A}(r)$. Applying $(*)$ gives

$$\begin{aligned} \widehat{A}(r) &= \int_{\mathbb{R}} A(t) e^{-2\pi i r t} dt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n \cap \{x: \langle x, \xi \rangle = t\}} 1_{[-\frac{1}{2}, \frac{1}{2}]^n}(x) dx \right) e^{-2\pi i r t} dt \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n \cap \{x: \langle x, \xi \rangle = t\}} \prod_{i=1}^n g(|x_i|) dx_1 \cdots dx_n \right) e^{-2\pi i r t} dt \\ &= \prod_{i=1}^n \int_{\mathbb{R}} g(|x_i|) e^{-2\pi i r x_i \xi_i} dx_i = \prod_{i=1}^n \frac{\sin \pi r \xi_i}{\pi r \xi_i}. \end{aligned}$$

Therefore, $A(t) = \int_{\mathbb{R}} e^{-2\pi i r t} \prod_{i=1}^n \frac{\sin \pi r \xi_i}{\pi r \xi_i} dr$. The integral on the right can now be optimized over the set of ξ_i with $\sum_i \xi_i^2 = 1$.

If any $\xi_i = \langle \xi, x_i \rangle$ is greater than $1/\sqrt{2}$, then

$$\begin{aligned} 1 &\geq \text{Proj}_{\{x_i=1/2\}} \left(\left[-\frac{1}{2}, \frac{1}{2} \right]^n \cap H \right) = \langle \xi, x_i \rangle \text{Area} \left(\left[-\frac{1}{2}, \frac{1}{2} \right]^n \cap H \right) \\ &= \xi_i \text{Area} \left(\left[-\frac{1}{2}, \frac{1}{2} \right]^n \cap H \right). \end{aligned}$$

So $\text{Area}([-\frac{1}{2}, \frac{1}{2}]^n \cap H) \leq \xi_i^{-1} \leq \sqrt{2}$. So, we may assume $\xi_i \leq 1/\sqrt{2}$. Let $p_i = \xi_i^{-2}$. Then $p_i \geq 2$ and $\sum p_i^{-1} = 1$. Apply Hölder's inequality (with n exponents) for $\sum_i \frac{1}{p_i} = 1$ to get

$$A(t) \leq \prod_{i=1}^n \left(\int_{\mathbb{R}} \left| \frac{\sin \pi r \xi_i}{\pi r \xi_i} \right|^{p_i} dr \right)^{\frac{1}{p_i}} = \prod_{i=1}^n \left(\frac{1}{\pi \xi_i} \int_{\mathbb{R}} \left| \frac{\sin r}{r} \right|^{p_i} dr \right)^{\frac{1}{p_i}}.$$

Finally, apply the following claim (that we will not prove): $\frac{1}{\pi} \int_{\mathbb{R}} \left| \frac{\sin t}{t} \right|^p dt \leq \sqrt{2/p}$ for $p \geq 2$, with equality if and only if $p = 2$. We get

$$A(t) \leq \prod_{i=1}^n \left(\frac{1}{\xi_i} \sqrt{\frac{1}{p_i}} \sqrt{2} \right)^{\frac{1}{p_i}} = \prod_{i=1}^n (\sqrt{2})^{\frac{1}{p_i}} = \sqrt{2}^{\sum_i \frac{1}{p_i}} = \sqrt{2},$$

as desired. □

Remark 2.2. Sharp Hausdorff-Young is not strong enough to prove our omitted claim.

Definition 2.3. $d\gamma_n(x) := \frac{1}{(2\pi)^{n/2}} e^{-(x_1^2 + \dots + x_n^2)/2} dx$

Theorem 2.4. (*Sharp Hausdorff-Young and Young's Inequalities* [Be]) Let $1 \leq p \leq 2$, $1/p + 1/p' = 1$. Define $A_p = \frac{p^{1/2p}}{(p')^{1/2p'}}$. For $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $\|\widehat{f}\|_{p'} \leq (A_p)^n \|f\|_p$.

Let $1/r = 1/p + 1/q - 1$, $1 \leq p, q, r \leq \infty$. Then $\|f * g\|_r \leq (A_p A_q / A_r)^n \|f\|_p \|g\|_q$, and both of these inequalities are sharp.

Proof: (Sketch) Define the Hermite polynomials $\{h_n(x)\}$ on \mathbb{R} via the generating function

$$e^{-\frac{t^2}{2} + xt} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n h_n(x).$$

For $g \in L_2(\mathbb{R}, d\gamma_1)$, write $g(x) = \sum_{n \geq 0} c_n h_n(x)$. Define

$$P_t g(x) = \sum_{n \geq 0} t^n c_n h_n(x).$$

We first need a version of Gaussian hypercontractivity (Thm. 5.5), where we use purely imaginary t . Unfortunately, Gross' proof of the Gaussian case fails for imaginary t . One can modify the proof for the case of the cube (Thm. 5.4) to handle imaginary t and then apply the Central Limit Theorem. A special case of the full hypercontractivity result is the following (due to Beckner):

$$\|P_{(i\sqrt{p-1})}(g)\|_{L_{p'}(d\gamma_1)} \leq \|g\|_{L_p(d\gamma_1)} \quad , 1 < p \leq 2. \quad (*)$$

Note that $|i\sqrt{p-1}| \leq 1$. Also, $P_{(i\sqrt{p-1})}$ takes a function from a lower L_p space into a higher one (hence the "hyper" in hypercontractivity). We use an explicit formula for the Mehler kernel $k_t(x, y)$ defined by $P_t(g)(x) = \int k_t(x, y)g(y)d\gamma_1(y)$:

$$k_t(x, y) = (1 - t^2)^{-\frac{1}{2}} \exp\left(-\frac{t^2(x^2 + y^2) - 2txy}{2(1 - t^2)}\right).$$

Writing out (*) gives $(\int |\int k_{(i\sqrt{p-1})}(x, y)g(y)d\gamma_1(y)|^{p'} d\gamma_1(x))^{1/p'} \leq (\int |g(x)|^p d\gamma_1(x))^{1/p}$. On the left, substitute $x = \sqrt{2\pi p'}u$, $y = \sqrt{2\pi p}v$, and on the right substitute $x = \sqrt{2\pi p}u$ to get

$$\left(\int \left|\int e^{2\pi iuv} g(\sqrt{2\pi p}v) e^{-\pi v^2} dv\right|^{p'}\right)^{1/p'} \leq A_p \left(\int |g(\sqrt{2\pi p}u)| e^{-\pi u^2}\right)^{1/p}.$$

So, sharp Hausdorff-Young holds for any f on \mathbb{R} of the form $f(v) = g(\sqrt{2\pi p}v)e^{-\pi v^2}$. By density, the theorem holds on \mathbb{R} for all f , as desired. Since the Fourier transform splits as a product of one-dimensional Fourier transforms, sharp Hausdorff-Young then holds for \mathbb{R}^n . (Sharpness follows by considering appropriate Gaussians.)

Sharp Young's inequality can be solved with spherical rearrangement techniques. We omit the details. ■

Problem 2.5. Find a "continuous" proof that $\|P_{(i\sqrt{p-1})}(g)\|_{p'} \leq \|g\|_p$, i.e. do not use results on the hypercube.

3. GROTHENDIECK INEQUALITIES

Theorem 3.1. (*Grothendieck's Inequality/Approximating the Cut Norm*, [AN])
 There is a randomized polynomial time algorithm such that, given an $n \times m$ real matrix $A = \{a_{ij}\}$ and unit vectors $x_i, y_j \in \mathbb{R}^{n+m}$, the algorithm finds $u_i, \nu_j \in \{-1, 1\}$ such that the expected value of the sum $\sum_{ij} a_{ij} u_i \nu_j$ is

$$\frac{2 \log(1 + \sqrt{2})}{\pi} \sum_{ij} a_{ij} \langle x_i, y_j \rangle.$$

Proof. (Rough sketch) It would be nice to “round” the x_i, y_j to signs in some way. That is, we would like to project x_i onto some random unit vector z , and then declare x_i to be $+1$ if x_i is on one side of the vector z , or -1 if x_i is on the other side of z . (And similarly for y_j .) This can almost be done with the following formula

$$\frac{\pi}{2} \mathbb{E}([\text{sign}\langle x, z \rangle] \cdot [\text{sign}\langle y, z \rangle]) = \sin^{-1}\langle x, y \rangle,$$

where z is uniform on the surface of the unit sphere. However, we want to get rid of the inverse sine term. We can do this, with a loss of a constant, by first mapping the vectors x_i, y_j into a vector of “quantum states”, and then by taking signs:

$$\frac{\pi}{2} \mathbb{E}([\text{sign}\langle x'_i, z \rangle] \cdot [\text{sign}\langle y'_j, z \rangle]) = (\log(1 + \sqrt{2})) \langle x_i, y_j \rangle, \quad \forall 1 \leq i \leq n, 1 \leq j \leq m.$$

Here $x'_i, y'_j \in \bigoplus_{k=0}^{\infty} (\mathbb{R}^{n+m})^{\otimes (2k+1)}$, $\|x'_i\|_2 = \|y'_j\|_2 = 1$. So, by linearity of expectation

$$\sum_{ij} a_{ij} \langle x_i, y_j \rangle = \frac{\pi}{2 \log(1 + \sqrt{2})} \mathbb{E} \left(\sum_{ij} a_{ij} [\text{sign}\langle x'_i, z \rangle] \cdot [\text{sign}\langle y'_j, z \rangle] \right).$$

Finally, define $u_i := \text{sign}\langle x'_i, z \rangle$ and $\nu_j := \text{sign}\langle y'_j, z \rangle$. □

Remark 3.2. This proof shows that the best constant K in Grothendieck's inequality (for an $n \times n$ real matrix) satisfies $K \leq \frac{\pi}{2 \log(1 + \sqrt{2})} \approx 1.7822139781913\dots$, where

$$\max_{\substack{x_1, \dots, x_n, y_1, \dots, y_n \in \ell_2 \\ \|x_i\|_2 = \|y_i\|_2 = 1}} \sum_{i,j=1}^n a_{ij} \langle x_i, y_j \rangle \leq K \max_{u_1, \dots, u_n, \nu_1, \dots, \nu_n \in \{-1, 1\}} \sum_{i,j=1}^n a_{ij} u_i \nu_j.$$

This proof can be generalized [BMMN] so that, instead of projecting onto a random line, one projects onto a random plane. One then concludes that $K \leq \frac{\pi}{2 \log(1 + \sqrt{2})} - \epsilon_0$ (for some effective constant ϵ_0).

Problem 3.3. What is the best possible rounding procedure in the proof of the above theorem? (This problem is also related to the operator P_{it} .)

In the following proof, we will combine many elements of the above. First, we restrict 3.1 to semidefinite $\{a_{ij}\}$. For the purposes of the theorem below, we may then assume that $x_i = y_i$. In this case, the sharp constant in the (positive semidefinite) Grothendieck inequality is known, and it is $\pi/2$. Since this version of the inequality is well understood, it is natural to generalize it in some sense. In this generalization, instead of rounding the vectors x_i to two different diametrically opposed vectors, consider rounding them to *any* k vectors. For simplicity, we state the theorem for $k = 4$.

Theorem 3.4. (Generalized Positive Semidefinite Grothendieck Inequality) (Theorem 3.1, 3.3, [KN2], Theorem 1.1, [KN1]) Let $A = \{a_{ij}\}$ be an $n \times n$ real symmetric positive semidefinite matrix. Let $\nu_1, \dots, \nu_4 \in \mathbb{R}^4$ be orthonormal vectors. Let $B = \{b_{ij}\} = \{\langle \nu_i, \nu_j \rangle\} = Id_4$ be the corresponding Gram matrix. Then

$$\max_{x_1, \dots, x_n \in S^{n-1}} \sum_{i,j=1}^n a_{ij} \langle x_i, x_j \rangle \leq \frac{1}{C(B)} \max_{\sigma: \{1,2,\dots,n\} \rightarrow \{1,2,3,4\}} \sum_{i,j=1}^n a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle,$$

where

$$C(B) = C(4-1, B) := \sup_{\substack{A_1, \dots, A_4, \cup_{i=1}^4 A_i = \mathbb{R}^3, \\ A_i \cap A_j = \emptyset, i \neq j}} \sum_{i=1}^4 \left\| \int_{A_i \cap \mathbb{R}^3} x d\gamma_3(x) \right\|^2.$$

Moreover, this inequality is sharp. That is, for the same inequality to hold for some constant K , it must be that $K \geq \frac{1}{C(B)}$.

Remark 3.5. $\int_{A_i} x d\gamma(x)$ is shorthand for the vector with entries $(\int_{A_i} x_1 d\gamma(x), \dots, \int_{A_i} x_n d\gamma(x))$.

Proof. (Rough sketch) We combine some randomness ideas from the Grothendieck inequality, and some Fourier analysis as in Hurwitz's proof of isoperimetry. Let $\{A_1, \dots, A_4\}$ be a partition of \mathbb{R}^3 . Using a $3 \times n$ matrix G of iid Gaussians, randomly "round" each x_i, x_j to some ν_i, ν_j . Specifically, let $\sigma: \{1, \dots, n\} \rightarrow \{1, 2, 3, 4\}$ so that $\sigma(i)$ is the unique $p \in \{1, 2, 3, 4\}$ with $Gx_i \in A_p$. So, by definition of expected value,

$$\mathbb{E} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle = \sum_{p,q=1}^4 \langle \nu_p, \nu_q \rangle \mathbb{P}[Gx_i \in A_p, Gx_j \in A_q].$$

Using yet another formula for the Mehler kernel allows the following Hermite-Fourier decomposition formula. For $x, y \in S^{n-1}$ and $E, F \subset \mathbb{R}^3$, we claim:

$$\mathbb{P}[Gx \in E, Gy \in F] = \gamma_3(E)\gamma_3(F) + \langle x, y \rangle \left\langle \int_E u d\gamma_3(u), \int_F u d\gamma_3(u) \right\rangle + \dots.$$

We then sum over i, j and combine these formulas. Positive semidefiniteness of a_{ij} allows us to throw out all but the second term in the latter formula. Then, taking a random assignment σ and taking the maximum over partitions as in the definition of $C(B)$ we get our desired result

$$\sum_{i,j=1}^n a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle \geq C(B) \sum_{i,j=1}^n a_{ij} \langle x_i, x_j \rangle.$$

□

4. THE PROPELLER CONJECTURE

Let C_1, C_2, C_3 be three disjoint cones in \mathbb{R}^2 with mutual basepoint the origin and mutual cone angle $2\pi/3$. The **Propeller Conjecture** says that $C(Id_4) = C(Id_3) = \frac{9}{8\pi}$, and the optimal A_i 's are given by $\{C_1 \times \mathbb{R}, C_2 \times \mathbb{R}, C_3 \times \mathbb{R}, \emptyset\}$. (Note that $C(Id_k) \leq C(Id_{k+1})$.)

In a forthcoming paper, Aukosh Jagannath, Assaf Naor and myself give a computer-assisted proof of the Propeller conjecture in dimension 3. There are two main theoretical reductions that occur, followed by a brute-force search of the zeros of some three-dimensional

function. (Note that $C(B)$ actually has five degrees of freedom, so there is some dimension reduction that allows the computation to become tractable.)

Step 1 is an explicit formula for $\int_{A_i} x d\gamma_3(x)$ (Proposition 4.2). (Khot and Naor already showed that the A_i are simplicial cones. Using polar coordinates, it suffices to consider four spherical triangles on S^2 with vertices $\{v_1, v_2, v_3, v_4\}$. We then compute $\sum_{i=1}^4 \|z_i\|_2^2$ with $z_i = \int_{A_i \cap S^2} x dS(x)$.) Step 2 involves a lot of algebra that eventually allows the elimination of two degrees of freedom. An interesting consequence, that helps with some quantitative estimates, is that, in some sense, $\sum_{i=1}^4 \|z_i\|_2^2$ has the same critical points as $\sum_{1 \leq i < j \leq 4} (\cos^{-1}(\langle v_i, v_j \rangle))^2$.

Note that $\sum_{1 \leq i < j \leq 4} (\cos^{-1}(\langle v_i, v_j \rangle))^2$ represents the squared lengths of six edges on the sphere. We think of these edges as rubber bands. To see how this claim helps our intuition, consider the following question: can all four vertices v_i lie in an open hemisphere and be in equilibrium under the rubber band forces? No, since the vertices “want” to shrink to a point. Considering different scenarios like this gives enough intuition to make some estimates that are useful in the numerical computation.

Problem 4.1. Is there a better way to go about this problem? We seem to have shown that the “direct approach” does not have any chance of working in higher dimensions.

Proposition 4.2. ([Br] p. 6) Suppose a spherical triangle $T \subset S^2$ has vertices $\{v_1, v_2, v_3\}$ and $\det(v_1, v_2, v_3) > 0$. Let dS denote Lebesgue measure on S^2 . Let $\theta_{ij} = \cos^{-1}(\langle v_i, v_j \rangle)$. Then the vector $z = \int_T x dS(x) \in \mathbb{R}^3$ satisfies

$$z = \frac{1}{2} \left(\theta_{12} \frac{v_1 \times v_2}{\|v_1 \times v_2\|_2} + \theta_{23} \frac{v_2 \times v_3}{\|v_2 \times v_3\|_2} + \theta_{31} \frac{v_3 \times v_1}{\|v_3 \times v_1\|_2} \right). \quad (4.1)$$

(Proof by picture) By duality, it suffices to compute $\langle z, v_1 \rangle$. Write $\langle z, v_1 \rangle = \int_T \langle x, v_1 \rangle dS$. Apply the Cauchy Projection Formula. One sees that $\langle z, v_1 \rangle$ is half the angle of a sector S , multiplied by the cosine of the angle between v_1 and a perpendicular to S . Thus,

$$\langle z, v_1 \rangle = \left\langle \frac{\theta_{23}}{2} \frac{v_2 \times v_3}{\|v_2 \times v_3\|_2}, v_1 \right\rangle = \theta_{23} \frac{\det(v_1, v_2, v_3)}{2 \sin \theta_{23}}.$$

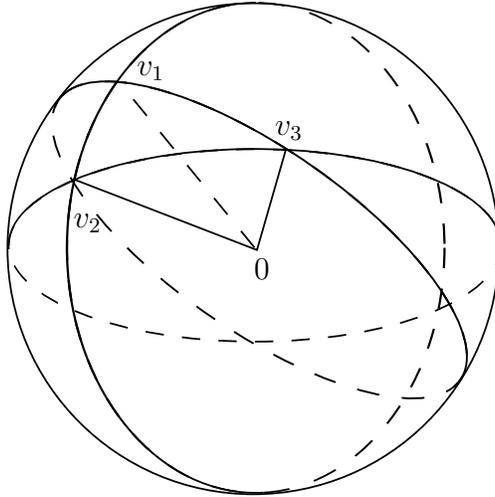


FIGURE 1. Spherical triangle, together with the origin and three sectors of discs.

5. APPENDIX

Theorem 5.1. (Jordan Curve Theorem) *A subspace of S^2 homeomorphic to S^1 separates S^2 into two simply-connected path-components.*

Theorem 5.2. (Dirichlet-Jordan Theorem) *Let f be of bounded variation on $[0, 1]$. Then the Fourier-Stieltjes series of f converges at each point to $\frac{1}{2}(f(x-) + f(x+))$ and it converges uniformly to $f(x)$ on any compact set on which the periodic extension of f is continuous. (In this theorem, we define the Fourier-Stieltjes series as $\widehat{\mu}(n) := \int_{(0,1]} e^{-2\pi inx} d\mu(x)$).*

Proof. (of Theorem 1.1) Let $\gamma(t) = (x(t), y(t))$, $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ denote the unit speed parametrization of Γ . Since γ is of bounded variation, it is differentiable almost everywhere (being the difference of two monotone functions). So, by assumption, $(x'(t))^2 + (y'(t))^2 = 1$ a.e., so $1 = \int_0^1 (x'(t))^2 + (y'(t))^2 dt$. Now,

$$\begin{aligned} A &= \left| \int_D dx dy \right| = \left| \int_D \frac{1}{2} d(xdy - ydx) \right| = \left| \frac{1}{2} \int_{\Gamma} xdy - ydx \right| \\ &= \frac{1}{2} \left| \int_0^1 (x(t)y'(t) - y(t)x'(t)) dt \right|, \end{aligned}$$

using Stokes. Using Dirichlet-Jordan (Thm. 5.2), write $x(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi int}$, $y(t) = \sum_{n \in \mathbb{Z}} b_n e^{2\pi int}$. By integrating by parts, we see that x', y' can be represented in $L_2(\mathbb{T})$ as $x'(t) \sim 2\pi i n a_n e^{2\pi int}$, $y'(t) \sim 2\pi i n b_n e^{2\pi int}$. Applying Parseval to our above identities yields

$$1 = \sum_{n \in \mathbb{Z}} 4\pi^2 n^2 (|a_n|^2 + |b_n|^2), \quad (*)$$

$$A = \frac{1}{2} \left| 2\pi i \sum_{n \in \mathbb{Z}} n (a_n \bar{b}_n + b_n \bar{a}_n) \right| \leq \pi \sum_{n \in \mathbb{Z}} |n| (|a_n|^2 + |b_n|^2), \quad (**)$$

using $|a\bar{b} + b\bar{a}| \leq 2|a||b| \leq |a|^2 + |b|^2$ from AMGM. So, plugging in $|n| \leq |n|^2$ in (**), then applying (*) to the result shows that $A \leq 1/4\pi$. Lastly, to get equality in the above argument, we must have $|n| (|a_n|^2 + |b_n|^2) = |n|^2 (|a_n|^2 + |b_n|^2)$, which forces $a_n = b_n = 0$ for $|n| > 1$, so that Γ is a circle. \square

Theorem 5.3. (Stationary Phase in Higher Dimensions) *Let $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$, $\psi: \mathbb{R}^n \rightarrow \mathbb{C}$. Define*

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x) dx.$$

Suppose $\phi(x_0) = 0$ and ϕ has a nondegenerate critical point at x_0 . (That is, the Hessian matrix $\partial^2\phi/\partial x_i\partial x_j$ is invertible at x_0 .) If ψ is supported in a sufficiently small neighborhood of x_0 , then as $\lambda \rightarrow \infty$,

$$I(\lambda) \sim \lambda^{-n/2} \sum_{j=0}^{\infty} a_j \lambda^{-j}.$$

Theorem 5.4. (Hypercontractivity on $\{0, 1\}^n$) *For $0 \leq \rho \leq 1$, $f: \{0, 1\}^n \rightarrow \mathbb{R}$ define*

$$T_\rho f(\omega) := \sum_{S \subseteq [n]} \rho^{|S|} \widehat{f}(S) W_S(\omega),$$

with $W_S(\omega) = (-1)^{\sum_{i \in S} \omega_i}$, $\widehat{f}(S) = \frac{1}{2^n} \sum_{\omega \in \{0,1\}^n} f(\omega) W_S(\omega)$. For $1 \leq p \leq q$, $0 \leq \rho \leq \sqrt{\frac{p-1}{q-1}}$,

$$\|T_\rho f\|_q \leq \|f\|_p.$$

Theorem 5.5. (Hypercontractivity on Gaussian space, Real t) Let $(\mathbb{R}^n, d\gamma_n)$ be the standard Gaussian probability space, and let P_t be the Ornstein-Uhlenbeck semigroup. Let $1 < p < q < \infty$, $e^{-t} \leq \sqrt{\frac{p-1}{q-1}}$. Then

$$\|P_t f\|_q \leq \|f\|_p.$$

Proof. (of Theorem 3.1)

We begin with two claims. The first is an exercise that reduces to a 2-dimensional computation, via rotation invariance.

Claim 1: Let x, y be unit vectors in a finite dimensional Hilbert space. If z is chosen randomly and uniformly according to the normalized Haar measure on the unit sphere of the space, then

$$\frac{\pi}{2} \mathbb{E}([\text{sign}\langle x, z \rangle] \cdot [\text{sign}\langle y, z \rangle]) = \sin^{-1}\langle x, y \rangle.$$

Claim 2: Let $\{x_i\}_{1 \leq i \leq n} \cup \{y_j\}_{1 \leq j \leq m}$ be unit vectors in a Hilbert space H . Define $c = \sinh^{-1}(1) = \log(1 + \sqrt{2})$. Then there is a set $\{x'_i\}_{1 \leq i \leq n} \cup \{y'_j\}_{1 \leq j \leq m}$ of unit vectors in some (finite dimensional) Hilbert space H' such that, if z is chosen randomly and uniformly on the unit sphere of H' , then

$$\frac{\pi}{2} \mathbb{E}([\text{sign}\langle x'_i, z \rangle] \cdot [\text{sign}\langle y'_j, z \rangle]) = c\langle x_i, y_j \rangle, \quad \forall 1 \leq i \leq n, 1 \leq j \leq m.$$

We now prove Claim 2. Using the Taylor expansion of \sin and the definition of the tensor product, we have

$$\sin(c\langle x, y \rangle) = \sum_{k=0}^{\infty} (-1)^k \frac{c^{2k+1}}{(2k+1)!} \langle x^{\otimes(2k+1)}, y^{\otimes(2k+1)} \rangle,$$

where $x^{\otimes \ell}$ denotes x tensored with itself ℓ times. Now, consider vectors $T(x), S(y)$ in the Hilbert space $\widetilde{H} := \bigoplus_{k=0}^{\infty} H^{\otimes(2k+1)}$ where the k^{th} coordinates are given by

$$T(x)_k = (-1)^k \sqrt{\frac{c^{2k+1}}{(2k+1)!}} x^{\otimes(2k+1)}, \quad S(y)_k = (-1)^k \sqrt{\frac{c^{2k+1}}{(2k+1)!}} y^{\otimes(2k+1)}.$$

Then $\sin(c\langle x, y \rangle) = \langle T(x), S(y) \rangle$, i.e. $c\langle x, y \rangle = \sin^{-1}\langle T(x), S(y) \rangle$. Moreover, by explicit calculation (using the definition of the norm in \widetilde{H}) we have $\|T(x)\|^2 = \sinh(c\|x\|^2)$, $\|S(y)\| = \sinh(c\|y\|^2)$.

Now, let $x'_i = T(x_i)$ and $y'_j = S(y_j)$. Since the x_i, y_i have unit length and $c = \sinh^{-1}(1)$, we see that x'_i, y'_j are also unit vectors (in \widetilde{H}). Define $H' = \text{span}\{x'_i, y'_j\}$, and note that H' has dimension $m + n$. Finally, let z be chosen uniformly on the unit ball of H' . Applying Claim 1 gives

$$\frac{\pi}{2} \mathbb{E}([\text{sign}\langle T(x_i), z \rangle] \cdot [\text{sign}\langle S(y_j), z \rangle]) = \sin^{-1}\langle T(x_i), S(y_j) \rangle = c\langle x_i, y_j \rangle,$$

using our expression for $c\langle x, y \rangle$ from above.

We are now ready to prove our theorem. Let x_i, y_j be unit vectors as above. Applying Claim 2, we find x'_i, y'_j as in the conclusion of the claim. Since x'_i, y'_j exist, there is a nonempty neighborhood containing them. Specifically, define $K \subset \bigoplus_{i=1}^{n+m} B_2^{n+m}$ as the ϵ -neighborhood of all such $\{x'_i, y'_j\}_{1 \leq i \leq n, 1 \leq j \leq m}$. Using Claim 1, our restrictions are of the form $\langle x'_i, y'_j \rangle = \sin(c \langle x_i, y_j \rangle)$. Our semidefinite constraints therefore give constraints on the Gram matrix of the $\{x'_i, y'_j\}_{ij}$. Up to appropriate errors, we can use the Ellipsoid method to find such points x'_i, y'_j in polynomial time. Thus (ignoring these errors) and adding our terms, we have

$$c \cdot \sum_{i,j} a_{ij} \langle x_i, y_j \rangle = \frac{\pi}{2} \mathbb{E} \left(\sum_{i,j} a_{ij} [\text{sign} \langle x'_i, z \rangle] \cdot [\text{sign} \langle y'_j, z \rangle] \right).$$

We conclude by choosing a random z and defining $u_i := \text{sign} \langle x'_i, z \rangle$ and $\nu_j := \text{sign} \langle y'_j, z \rangle$. \square

Proof. (of Theorem 3.4)

The idea of the proof is, in some sense, a combination of Theorems 1.1 and 3.1.

First, by some preliminary calculations (which we omit), the supremum in the definition of $C(B)$ is attained for some partition of \mathbb{R}^n into disjoint measurable sets $\mathbb{R}^n = \cup_{i=1}^{k-1} A_i$.

Now, let $\{g_{ij}\}_{i \in [m], j \in [n]}$ be iid standard Gaussian random variables, and let $G = \{g_{ij}\}$ be the corresponding $m \times n$ random Gaussian matrix, with $m = k - 1$. Define a random assignment $\sigma: [n] \rightarrow [k]$ by setting $\sigma(i)$ to be the unique $p \in \{1, \dots, k\}$ such that $Gx_i \in A_p$. Then, for every $i, j \in [n]$ we have

$$\begin{aligned} \mathbb{E} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle &= \sum_{p,q=1}^k \langle \nu_p, \nu_q \rangle \mathbb{P} [Gx_i \in A_p, Gx_j \in A_q] \\ &= \sum_{p,q=1}^k b_{pq} \mathbb{P} [Gx_i \in A_p, Gx_j \in A_q], \quad (*) \end{aligned}$$

by definition of $b_{p,q}$. We now use (Hermite) Fourier analysis to find an expression for the stated probability. For $x, y \in S^{n-1}$ and $E, F \subset \mathbb{R}^m$, we claim:

$$\begin{aligned} \mathbb{P} [Gx \in E, Gy \in F] &= \gamma_m(E) \gamma_m(F) + \langle x, y \rangle \left\langle \int_E u d\gamma_m(u), \int_F u d\gamma_m(u) \right\rangle \\ &\quad + \sum_{\ell=2}^{\infty} \langle x^{\otimes \ell}, y^{\otimes \ell} \rangle \sum_{\substack{s \in (\mathbb{Z}_{\geq 0})^m \\ s_1 + \dots + s_m = \ell}} \alpha_s(E) \alpha_s(F) \quad , \alpha_s(E), \alpha_s(F) \in \mathbb{R}. \end{aligned}$$

To prove this claim, let $r = \langle x, y \rangle$, and let g, h be iid standard Gaussian random variables, and let g_1, \dots, g_m be iid standard Gaussian random vectors in \mathbb{R}^n . For a fixed $i \in \{1, \dots, m\}$, rotation invariance says that $(\langle g_i, x \rangle, \langle g_i, y \rangle) \in \mathbb{R}^2$ is equal in distribution with $(g, rg + \sqrt{1-r^2}h)$. So, we see that the covariance matrix of $(\langle g_i, x \rangle, \langle g_i, y \rangle)$ is $\Gamma = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ with

$\Gamma^{-1} = \frac{1}{1-r^2} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}$, so the density is

$$f_r(u, \nu) := \frac{1}{2\pi\sqrt{1-r^2}} \exp \left(-\frac{u^2 - 2ru\nu + \nu^2}{2(1-r^2)} \right)$$

Now, define the Hermite polynomials $\{H_k\}_{k=0}^\infty$ via the formula

$$H_k(t) := (-1)^k e^{t^2} \frac{d^k}{dt^k} (e^{-t^2}) = \sum_{s=0}^{\lfloor k/2 \rfloor} \frac{(-1)^s k!}{s!(k-2s)!} (2t)^{k-2s}$$

Applying a formula for the Poisson kernel for Hermite polynomials (or Mehler kernel) shows that

$$f_r(u, \nu) = \frac{e^{-(u^2+\nu^2)/2}}{2\pi} \sum_{k \geq 0} \frac{r^k}{2^k k!} H_k\left(\frac{u}{\sqrt{2}}\right) H_k\left(\frac{\nu}{\sqrt{2}}\right)$$

Now, since $(Gx, Gy) \in \mathbb{R}^{2m}$ has the same distribution as the vector $\{(\langle g_i, x \rangle, \langle g_i, y \rangle)\}_{i=1}^m$, whose planar entries are iid with density f_r we see that

$$\mathbb{P}[Gx \in E, Gy \in F] = \int_{E \times F} \left(\prod_{i=1}^m f_r(u_i, \nu_i) \right) dud\nu$$

So, inserting the above expression for f_r and re-arranging things gives our claim.

We now take equation (*) and substitute in our claim. Now, using that A, B are positive semidefinite allows us to throw away several terms. For example, the classical version of positive semidefiniteness shows us that $\sum_{i,j} a_{ij} \alpha_i \alpha_j = \alpha^t A \alpha \geq 0$. Similarly, a tensorization trick shows us that

$$\sum_{i,j} a_{ij} \langle x_i^{\otimes \ell}, x_j^{\otimes \ell} \rangle = \sum_{i,j} \langle u_i, u_j \rangle \langle x_i^{\otimes \ell}, x_j^{\otimes \ell} \rangle = \left\| \sum_i u_i \otimes (x_i^{\otimes \ell}) \right\|_2^2 \geq 0$$

Thus, using the positive semidefiniteness of A and B , allows us to throw away all but the first order term of the claim, yielding

$$\mathbb{E} \sum_{i,j=1}^n \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle \geq \sum_{i,j=1}^n a_{ij} \langle x_i, x_j \rangle \sum_{p,q=1}^k b_{pq} \left\langle \int_{A_p} x d\gamma_{k-1}(x), \int_{A_q} x d\gamma_{k-1}(x) \right\rangle$$

So, using our choice of the A_i , and the definition of $C(B)$, there must exist an assignment $\sigma: [n] \rightarrow [k]$ such that

$$\sum_{i,j=1}^n a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle \geq C(B) \sum_{i,j=1}^n a_{ij} \langle x_i, x_j \rangle$$

as desired. ■

Showing the required sharpness is fairly tedious, so we omit the calculation. By using properties of Gaussians and a few tricks, one can show that

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} \langle x, y \rangle \cdot \left\langle \frac{x}{\|x\|_2}, \frac{y}{\|y\|_2} \right\rangle d\gamma_m(x) d\gamma_m(y) \geq 1 - O(1/m)$$

One can then approximate this integral by a finite sum, apply the inequality

$$\max_{x_1, \dots, x_n \in S^{n-1}} \sum_{i,j=1}^n a_{ij} \langle x_i, x_j \rangle \leq K \max_{\sigma: [n] \rightarrow [k]} \sum_{i,j=1}^n a_{ij} \langle \nu_{\sigma(i)}, \nu_{\sigma(j)} \rangle$$

which shows that (after a few additional arguments) we have $1 \leq KC(B)$, as desired. □

Proof. (Of Proposition 4.2) Using the definition of z_4 , we have

$$\langle z_4, v_1 \rangle = \int_{T_4} \langle v_1, x \rangle dS(x) \quad (5.1)$$

We now apply the Cauchy projection formula. Define $\Pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ so that $\Pi(x)$ is the orthogonal projection of x onto the unique plane that is intersecting the origin and perpendicular to v_1 . The projection formula says

$$\int_{T_4} \langle v_1, x \rangle dS(x) = \text{Area}_{\mathbb{R}^2}[(\Pi(T_4)) \cap \{x: \langle v_1, x \rangle \geq 0\}] - \text{Area}_{\mathbb{R}^2}[(\Pi(T_4)) \cap \{x: \langle v_1, x \rangle < 0\}] \quad (5.2)$$

Consider the convex hull $H \subset \mathbb{R}^3$ of the spherical triangle T_4 together with the origin. The boundary of H consists of T_4 and three sectors of discs (see Figure 1). By definition of Π , if we apply Π to these three sectors, only one of them has nonzero projected \mathbb{R}^2 area. Call this sector S , and note that the spherical geodesic joining v_2 to v_3 is contained in S . (This geodesic is unique since $\theta_{23} < \pi$.) By choice of S ,

$$\text{Area}_{\mathbb{R}^2}[(\Pi(T_4)) \cap \{x: \langle v_1, x \rangle \geq 0\}] - \text{Area}_{\mathbb{R}^2}[(\Pi(T_4)) \cap \{x: \langle v_1, x \rangle < 0\}] = \text{Area}_{\mathbb{R}^2}(\Pi(S)) \quad (5.3)$$

Finally, observe that the right side of (5.3) can be computed explicitly. The sector S is a planar region, tilted at some angle (relative to v_1), and then projected perpendicular to v_1 . Therefore $\text{Area}(\Pi(S))$ is half the angle of the sector S , multiplied by the cosine of the angle between v_1 and a perpendicular to S . (We use the perpendicular $v_2 \times v_3$, which has positive inner product with v_1 by assumption.) Consequently

$$\text{Area}_{\mathbb{R}^2}(\Pi(S)) = \left\langle \frac{1}{2} \theta_{23} \frac{v_2 \times v_3}{\|v_2 \times v_3\|_2}, v_1 \right\rangle \quad (5.4)$$

Combining (5.1),(5.2),(5.3) and (5.4) yields

$$\langle z_4, v_1 \rangle = \theta_{23} \frac{\det(v_1, v_2, v_3)}{2 \sin \theta_{23}} = \frac{\theta_{23} \det(v_1, v_2, v_3)}{2 \sin \theta_{23}} \quad (5.5)$$

Finally, (4.1) follows from the above by duality. That is, if we use the biorthogonal dual basis $\{v_1^*, v_2^*, v_3^*\}$, we may write

$$\begin{aligned} z_4 &= \langle z_4, v_1 \rangle v_1^* + \langle z_4, v_2 \rangle v_2^* + \langle z_4, v_3 \rangle v_3^* \\ &= \frac{\langle z_4, v_1 \rangle v_2 \times v_3 + \langle z_4, v_2 \rangle v_3 \times v_1 + \langle z_4, v_3 \rangle v_1 \times v_2}{\det(v_1, v_2, v_3)}, \text{ by Cramer's rule} \end{aligned}$$

Applying (5.5) three times, with its indices permuted appropriately, gives (4.1). □

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