ISOPERIMETRY AND THE ORNSTEIN-UHLENBECK OPERATOR

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Abstract. We first survey the relation between the classical isoperimetric problem, the isoperimetric problem for the Gaussian measure, and the Ornstein-Uhlenbeck operator. We then describe a generalization of these results, which was posed by Isaksson and Mossel [10]. Some results on the conjecture of Isaksson and Mossel [7] will then be described. Both probabilistic and analytic methods will be emphasized. Finally, we describe applications to theoretical computer science.

1. Euclidean Isoperimetry

Let $n \geq 1$ be an integer, let $A \subseteq \mathbb{R}^n$ be a Borel set with smooth boundary ∂A. Let $\mathrm{vol}_n(A)$ denote the Euclidean volume of A, and let $\mathrm{vol}_{n-1}(\partial A)$ denote the Euclidean area of the boundary of A.

Theorem 1.1 (**Euclidean Isoperimetric Inequality**). *The Euclidean ball has the smallest boundary among all sets of fixed volume. That is, let $B \subseteq \mathbb{R}^n$ be a Euclidean ball such that $\mathrm{vol}_n(A) = \mathrm{vol}_n(B)$. Then*

$$\mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B).$$

One way of proving this theorem uses **symmetrization**. That is, given any $A \subseteq \mathbb{R}^n$, we rearrange A into a more symmetric set with smaller surface area but identical volume. If $A \subseteq \mathbb{R}^2$, we take any line L that passes through A, and then write A as a union of one-dimensional sets that are perpendicular to L. Then, slide each such one-dimensional set such that it is a line segment symmetric with respect to reflection across L. The rearranged set has the same volume as A. Intuitively, the rearranged set also has smaller surface area, since the surface area at a point on the boundary of A becomes averaged after rearrangement.

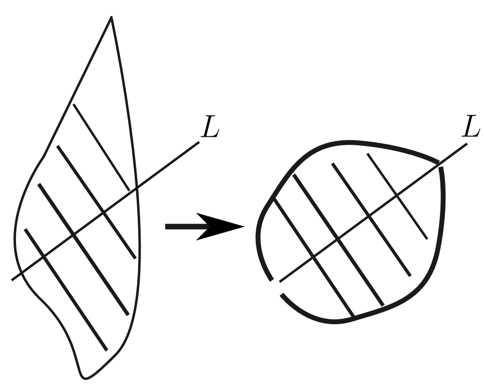

Figure 1. Classical Symmetrization

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Remark 1.2. A posteriori, since there exists a way to (non-strictly) decrease the perimeter of any given set while preserving its volume, we think of the classical isoperimetric problem as a convex (or concave) optimization problem. Note that, the more symmetric a set becomes, the more closely the set resembles a Euclidean ball.

Definition 1.3. Let $n \geq 1$, $n \in \mathbb{Z}$, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, let $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$, and let $||x||_2 := \langle x, x \rangle^{1/2}$. Let dx denote Lebesgue measure on $\mathbb{R}^n$, and define $d\gamma_n(x) := e^{-||x||_2^2/2}(2\pi)^{-n/2}dx$. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a bounded function and let $t \geq 0$. Let $\Delta := -\sum_{i=1}^n \partial^2/\partial x_i^2$. Let $T_t f(x)$ denote the **classical heat semigroup** applied to f. That is, for $x \in \mathbb{R}^n$,

$$e^{-t\Delta} f(x) := \int_{\mathbb{R}^n} f(x + y\sqrt{2t}) d\gamma_n(y) = \mathbb{E} f(x + Y\sqrt{2t}), \quad Y \sim N(0, 1).$$

The classical isoperimetric inequality is a consequence of the following inequality for the heat semigroup $e^{-t\Delta}$, which can be proven e.g. by symmetrization [14, 2].

Theorem 1.4 (Heat Semigroup Isoperimetric Inequality). *Let $A \subseteq \mathbb{R}^n$, and let $B \subseteq \mathbb{R}^n$ be a Euclidean ball such that $\mathrm{vol}_n(A) = \mathrm{vol}_n(B)$. Then, for all $t \geq 0$,*

$$\int_{\mathbb{R}^n} 1_A e^{-t\Delta} 1_A dx \leq \int_{\mathbb{R}^n} 1_B e^{-t\Delta} 1_B dx.$$

Remark 1.5. Theorem 1.4 allows a reinterpretation of Theorem 1.1. As heat flows out of a given set, the one that preserves the most of its heat is the ball. Also, to get Theorem 1.1 from Theorem 1.4, note that $\int_{\mathbb{R}^n} (1 - 1_A) e^{-t\Delta} 1_A dx$ measures the perimeter of a smooth set A as $t \to 0$ [14]. Then use $\int_{\mathbb{R}^n} (1 - 1_A) e^{-t\Delta} 1_A dx = \mathrm{vol}_n(A) - \int_{\mathbb{R}^n} 1_A e^{-t\Delta} 1_A dx$.

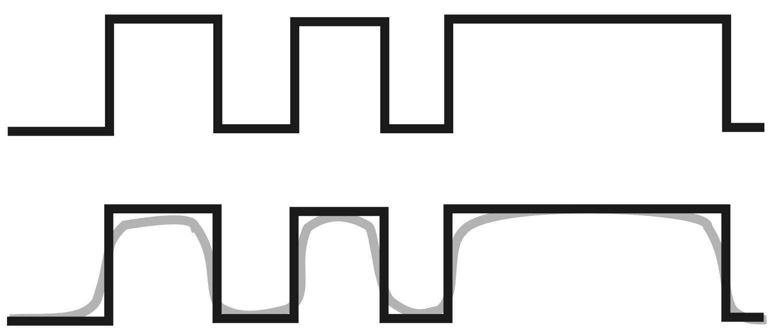

FIGURE 2. Depiction of heat evolution of the indicator function of a set

Remark 1.6. To symmetrize a function f, let $f \colon \mathbb{R}^n \to \mathbb{R}$, and write f as follows.

$$f(x) = \int_0^\infty 1_{\{x \in \mathbb{R}^n \colon f(x) > t\}}(t) dt$$

Then, perform symmetrizations on the super level sets $\{x \in \mathbb{R}^n \colon f(x) > t\}$.

Theorem 1.7 (Double Bubble Theorem [9, 16]). *Let $A_1, A_2 \subseteq \mathbb{R}^n$ be two sets of fixed volumes a_1, a_2. Let B_1, B_2 be a standard double bubble where $\mathrm{vol}_n(B_1) = \mathrm{vol}_n(A_1)$ and $\mathrm{vol}_n(B_2) = \mathrm{vol}_n(A_2)$. That is, ∂B_1 consists of two spherical caps, ∂B_2 consists of two spherical caps, and $\partial B_1 \cup \partial B_2$ is three spherical caps that meet at 120 degree angles. Then*

$$\mathrm{vol}_{n-1}(\partial A_1 \cup \partial A_2) \geq \mathrm{vol}_{n-1}(\partial B_1 \cup \partial B_2).$$

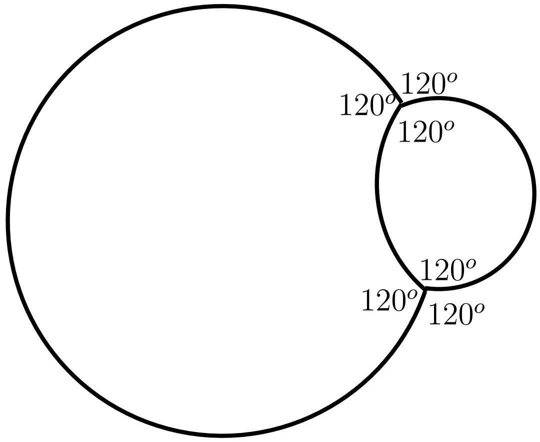

FIGURE 3. A Standard Double Bubble in $\mathbb{R}^2$

Problem 1.8 (**Multi Bubble Problem**)**.** Let $k \geq 3$, $k \in \mathbb{Z}$. Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ be sets of fixed volume. Try to minimize the surface area $\mathrm{vol}_{n-1}(\partial A_1 \cup \cdots \cup \partial A_k)$.

Remark 1.9. Essentially nothing is known about Problem 1.8. Whereas symmetrization methods aid the proof of Theorem 1.7, these methods seem to say nothing about the Multi Bubble Problem. So, a lack of applicability of symmetrization stifles our progress.

2. Gaussian Isoperimetry

We now observe how the results of the previous section change when Lebesgue measure is replaced with the Gaussian measure.

Definition 2.1. Let $z \in \mathbb{R}^n$, $t \in \mathbb{R}$. Define $H := \{x \in \mathbb{R}^n : \langle x, z \rangle \geq t\}$. We call $H \subseteq \mathbb{R}^n$ a **half space**.

Theorem 2.2 (**Gaussian Isoperimetric Inequality**)**.** *The half space has the smallest Gaussian boundary among all sets of fixed Gaussian volume. That is, let $H \subseteq \mathbb{R}^n$ be a half space such that $\gamma_n(A) = \gamma_n(H)$.*

$$\gamma_{n-1}(\partial A) \geq \gamma_{n-1}(\partial H).$$

Recall, $\gamma_{n-1}(\partial A) := \liminf_{\delta \to 0^+} (1/2\delta)\gamma_n(\{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^n \text{ such that } ||x - y||_2 \leq \delta\})$.

For applications to theoretical computer science, it is more useful to have a Gaussian version of Theorem 1.4, the semigroup isoperimetric inequality. To this end, we first replace the heat semigroup by the **Ornstein-Uhlenbeck semigroup**. For smooth $f : \mathbb{R}^n \to \mathbb{R}$, and $x \in \mathbb{R}^n$, define $Lf(x) := \langle x, \nabla f(x) \rangle + \Delta f(x)$.

Definition 2.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. For $x \in \mathbb{R}^n$ and $t \geq 0$, define

$$e^{-tL} f(x) := \int_{\mathbb{R}^n} f(xe^{-t} + y\sqrt{1 - e^{-2t}}) d\gamma_n(y) = \mathbb{E} f(xe^{-t} + Y\sqrt{1 - e^{-2t}}), \quad Y \sim N(0, 1).$$

Theorem 2.4 (**Ornstein-Uhlenbeck Semigroup Isoperimetric Inequality**, [1, 3])**.** *Let $A \subseteq \mathbb{R}^n$ and let $H \subseteq \mathbb{R}^n$ be a half-space such that $\gamma_n(A) = \gamma_n(H)$. Then, for all $t \geq 0$,*

$$\int_{\mathbb{R}^n} 1_A e^{-tL} 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_H e^{-tL} 1_H d\gamma_n.$$

As in Theorem 1.4, this inequality can be proven by symmetrization methods.

We now state a Gaussian version of the Multi Bubble Problem, Problem 1.8.

Definition 2.5. Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ be measurable. We say that $\{A_i\}_{i=1}^k$ is a **partition** of $\mathbb{R}^n$ if $\cup_{i=1}^k A_i = \mathbb{R}^n$, and $\gamma_n(A_i \cap A_j) = 0$ for all $i, j \in \{1, \ldots, k\}$, $i \neq j$.

Conjecture 1 (**Standard Simplex Conjecture**, [10]). *Let $n \geq 2$, and let $3 \leq k \leq n+1$. Let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$. Suppose $\gamma_n(A_i) = 1/k$ for all $i \in \{1, \ldots, k\}$. Let $\{z_i\}_{i=1}^k$ be the vertices of a regular simplex centered at the origin of $\mathbb{R}^n$. For all $i \in \{1, \ldots, k\}$, define $B_i := \{x \in \mathbb{R}^n : \langle x, z_i \rangle = \max_{j \in \{1,\ldots,k\}} \langle x, z_j \rangle\}$. Then for all $t \geq 0$,*

$$\sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} e^{-tL} 1_{A_i} d\gamma_n \leq \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{B_i} e^{-tL} 1_{B_i} d\gamma_n$$

Remark 2.6. Symmetrization does not seem to solve Conjecture 1.

Remark 2.7. The first variation condition for an optimal set in Conjecture 1 has the following intuitive interpretation. Suppose we have k regions of different immiscible metals, i.e. metals that do not mix with each other. Suppose we then heat the metals to allow their boundaries to settle into stable positions. Then these regions are optimal for Conjecture 1 if they do not move during this heating process [5].

Theorem 2.8. [7] *Let $k = 3$, $n \geq 2$. Then Conjecture 1 holds for $t(n) < t < \infty$.*

Our methods use mostly elementary real analysis, and they rely on the analytic and geometric aspects of the problem. As $t \to \infty$, we relate Conjecture 1 to previous work in maximizing the sum of squared Fourier coefficients of partitions [12, 13, 8].

Remark 2.9. As $t \to 0$, the Standard Simplex Conjecture approaches a Gaussian version of the Multi Bubble problem. The $k = 3$ case of this problem was solved in [4]. Their proof reduces the triple bubble problem in Gaussian space to the double bubble problem on the sphere. With this reduction, symmetrization methods then apply on the sphere. Their reduction uses the fact that, if we have three sets A_1, A_2, A_3 that partition $\mathbb{R}^n$, then the total perimeter of these three sets is the same as the total perimeter of any two of the sets. However, a similar reduction does not seem to be available for the Standard Simplex Conjecture for $k = 3$, since for $t > 0$, the quantity (1) does not exactly measure the perimeter of the three sets. And it is unclear how to compare the quantity (1) for three sets with the same quantity for two sets.

Remark 2.10. There are a few reasons that the Gaussian multi-bubble problem (Conjecture 1 as $t \to 0$) and the general Conjecture 1 for $t > 0$ are similar. However there are a few reasons that these problems are totally different. For Multi Bubble problems, existence and regularity of optimal sets require a deep result of Almgren, and the main difficulty lies in showing that the optimal sets are connected. However, for Conjecture 1 for t near ∞, existence of optimal sets and connectedness are relatively easy.

2.1. **Comments on the Proof of Theorem 2.8.** For $\rho \in (-1, 1)$, and $f : \mathbb{R}^n \to [0, 1]$, define the (non-semigroup) version of the Ornstein-Uhlenbeck operator by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y).$$

Instead of proving the Standard Simplex Conjecture directly, we instead try to maximize

$$\sum_{i=1}^{k}\int 1_{A_i}\frac{d}{d\rho}T_\rho 1_{A_i}d\gamma_n. \qquad (*)$$

To see why this is desirable, we need to use Hermite-Fourier analysis. Recall that the Hermite polynomials $h_\ell(x)$, $x\in\mathbb{R}$ are defined by

$$e^{\lambda x-\lambda^2/2}=\sum_{\ell\in\mathbb{Z}_{\geq 0}}\lambda^\ell h_\ell(x).$$

Also, $\{h_\ell\sqrt{\ell!}\}_{\ell\in\mathbb{Z}_{\geq 0}}$ is an orthonormal basis of $L_2(\gamma_1)$. For $\ell=(\ell_1,\ldots,\ell_n)\in\mathbb{Z}_{\geq 0}^n$, define $|\ell|:=\sum_{i=1}^n\ell_i$, $\ell!:=\prod_{i=1}^n\ell_i!$, and for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, define $h_\ell(x):=\prod_{i=1}^n h_{\ell_i}(x_i)$. Recall that

$$T_\rho h_\ell(x)=\rho^{|\ell|}h_\ell(x).$$

So,

$$T_\rho 1_{A_i}(x)=\sum_{\ell\in\mathbb{N}^n}\rho^{|\ell|}\sqrt{\ell!}\,h_\ell(x)\Big(\int_{A_i}\sqrt{\ell!}\,h_\ell(y)d\gamma_n(y)\Big) \qquad (**)$$

Now, the first variation condition for optimizing $(*)$ over partitions into $k=3$ sets with no measure restriction says that: for sets with smooth boundary, $(\partial A_i)\cap(\partial A_j)$ is contained in the set where $(d/d\rho)T_\rho(1_{A_i}-1_{A_j})=0$. Now, if we take the derivative in ρ of $(**)$, we get rid of the $\ell=0$ term in $(**)$, which is $\gamma_n(A_i)$. The Standard Simplex Conjecture has a measure restriction, which is annoying, and it is nice to get rid of the measure in $(**)$.

Now, the special thing about $k=3$ sets is that we expect the measure restriction to not matter for small $\rho>0$. And this turns out to be the case. That is, if we try to maximize $(*)$ over partitions into $k=3$ sets without any measure restriction, we expect the optimal sets to be the same as in the case that we do use a measure restriction $\gamma_n(A_i)=1/3$ for $i=1,2,3$.

Okay, so let's now see why we have some hope for using the first variation condition for maximizing problem $(*)$ with no measure restriction. Differentiating $(**)$ in ρ and removing the linear term, the remaining terms are morally bounded by $O(\rho^2)$. So, if the optimal partition $\{A_1,A_2,A_3\}$ is ρ-close to a standard simplex, then the linear term in $(**)$ dominates the other terms to scales $O(\rho^2)$. So, we can deduce that actually the optimal partition is ρ^2-close to the standard simplex. And now we can morally iterate this procedure. Making this procedure precise becomes very difficult, but it can be done. Controlling the errors requires very precise geometric control of the higher order terms in $(**)$. A naive approach to such an error bound is to bound each Hermite polynomial separately, and then sum them all up. However, this procedure terminates the iterative procedure after finitely many steps. So, we require more precise control, which comes by estimating the Gaussian heat kernel directly via integration by parts in specific geometric regions.

Surprisingly, this strategy fails for $\rho<0$.

3. Two Applications of Conjecture 1

Theorem 3.4 below represents the complexity theoretic motivation of Conjecture 1. Informally, if Conjecture 1 is true, then we know the best way to approximate solutions of the

MAX-k-CUT problem. This problem tries to find the partition of a given graph into k pieces that cuts the most edges of the graph.

Also, Conjecture 1 implies the Plurality is Stablest Conjecture, Conjecture 2 below. In this conjecture, we have an election between k candidates where a large number of people have cast their votes. From the votes, we want to determine the winner of the election in a way that is most stable to tabulation mistakes or other potentially random changes in the votes. We also assume that no single vote has too much influence over the outcome of the election. Then Conjecture 2 says that the best way of choosing the winner of the election is to take the plurality.

3.1. **The MAX-k-CUT Problem.**

Definition 3.1 (**MAX-k-CUT**). Let $k \in \mathbb{N}$, $k \geq 2$. We define the weighted MAX-k-CUT problem. In this problem, we are given a finite graph, defined by a vertex set V and an edge set $E \subseteq V \times V$. We are also given a weight function $w\colon E \to [0,1]$. A k-cut is a function $c\colon V \to \{1,\ldots,k\}$. The goal of the MAX-k-CUT problem is to find the following quantity:

$$\max_{c\colon V\to\{1,\ldots,k\}} \sum_{\substack{(i,j)\in E\colon\\ c(i)\neq c(j)}} w(i,j).$$

Definition 3.2 (**Γ-MAX-2LIN(k)**). Let $k \in \mathbb{N}$, $k \geq 2$. We define the Γ-MAX-2LIN(k) problem. In this problem, we are given $m \in \mathbb{N}$ and $2m$ variables $x_i \in \mathbb{Z}/k\mathbb{Z}$, $i \in \{1,\ldots,2m\}$. We are also given a set $E \subseteq \{1,\ldots,2m\}\times\{1,\ldots,2m\}$ of cardinality m. An element $(i,j) \in E$ corresponds to one of m linear equations of the form $x_i - x_j = c_{ij} (\mathrm{mod}\, k)$, $i,j \in \{1,\ldots,2m\}$, $c_{ij} \in \mathbb{Z}/k\mathbb{Z}$. We are also given a weight function $w\colon E \to [0,1]$. The goal of the Γ-MAX-2LIN(k) problem is to find the following quantity:

$$\max_{(x_1,\ldots,x_{2m})\in(\mathbb{Z}/k\mathbb{Z})^{2m}} \sum_{\substack{(i,j)\in E\colon\\ x_i-x_j=c_{ij}(\mathrm{mod}\, k)}} w(i,j).$$

Definition 3.3 (**Unique Games Conjecture**, [11]). For every $\varepsilon \in (0,1)$, there exists a prime number $p(\varepsilon)$ such that no polynomial time algorithm can distinguish between the following two cases, for instances of Γ-MAX-2LIN($p(\varepsilon)$) with $w = 1$:

(i) (3.2) is larger than $(1-\varepsilon)m$, or
(ii) (3.2) is smaller than εm.

Theorem 3.4. (**Optimal Approximation for MAX-k-CUT**, [10][Theorem 1.13],[6]). *Let $k \in \mathbb{N}$, $k \geq 2$. Let $\{A_i\}_{i=1}^{k} \subseteq \mathbb{R}^{k-1}$ be a regular simplicial conical partition. Define*

$$\alpha_k := \inf_{-\frac{1}{k-1}\leq\rho\leq 1} \frac{k - k^2\sum_{i=1}^{k}\int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n}{(k-1)(1-\rho)} = \inf_{-\frac{1}{k-1}\leq\rho\leq 0} \frac{k - k^2\sum_{i=1}^{k}\int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n}{(k-1)(1-\rho)}.$$

Assume Conjecture 1 and the Unique Games Conjecture. Then, for any $\varepsilon > 0$, there exists a polynomial time algorithm that approximates MAX-k-CUT within a multiplicative factor $\alpha_k - \varepsilon$, and it is NP-hard to approximate MAX-k-CUT within a multiplicative factor of $\alpha_k + \varepsilon$.

The connection between Conjecture 1 and Theorem 3.4 comes from the semidefinite programming based algorithm for approximately solving MAX-k-CUT. Let $G = (V,E)$ be a graph on n vertices. Let $\{z_i\}_{i=1}^k$ be the vertices of a regular simplex in $\mathbb{R}^{k-1}$ centered at the origin. We can then rewrite the MAX-k-CUT problem as follows

$$\max_{c\colon V\to\{z_1,\ldots,z_k\}} \frac{k-1}{k} \sum_{\substack{(i,j)\in E\colon\\ c(i)\neq c(j)}} w(i,j)(1-\langle c(i),c(j)\rangle). \qquad (*)$$

We now consider a more general optimization problem, where we allow c to take any value on S^{n-1}, but we now make linear constraints on the inner products $\langle c(i),c(j)\rangle$. In particular, we try to find the following quantity, which is guaranteed to be larger than $(*)$:

$$\max_{\substack{c\colon V\to\mathbb{R}^n\,:\,\langle c(i),c(i)\rangle=1\,\forall\, i\in V,\\ \langle c(i),c(j)\rangle\geq -1/(k-1)\,\forall\, i,j\in V}} \sum_{(i,j)\in E} w(i,j)(1-\langle c(i),c(j)\rangle). \qquad (**)$$

We can compute $(**)$ in polynomial time in n, using semidefinite programming. Note that $(**)$ is automatically larger than $(*)$. Surprisingly, $(**)$ is not that much larger than $(*)$. To show this, consider c which achieves the optimum value in $(**)$, or nearly so. We then "round" $c(i)$ for each $i=1,\ldots,n$ to some z_j with $j\in\{1,\ldots,k\}$. And the best way to do this rounding is exactly given by using the partition from Conjecture 1. To see this, let m be a positive integer, and let $\{A_1,\ldots,A_k\}$ be a partition of $\mathbb{R}^m$. Let G be an $m\times n$ matrix of iid Gaussian variables. And for each $i=1,\ldots,n$, define $C(i):=z_j$ if and only if $Gc(i)\in A_j$. We then compute the expected value of the k-cut that result from this randomized rounding. To compare the result to the larger quantity $(**)$, we also divide by $(**)$:

$$\begin{aligned}
&\frac{\mathbb{E}_G \sum_{(i,j)\in E\colon C(i)\neq C(j)} w(i,j)}{\frac{k-1}{k}\sum_{\substack{(i,j)\in E\colon\\ c(i)\neq c(j)}} w(i,j)(1-\langle c(i),c(j)\rangle)} = \frac{\sum_{(i,j)\in E\colon} w(i,j)\mathbb{P}(C(i)\neq C(j))}{\frac{k-1}{k}\sum_{\substack{(i,j)\in E\colon\\ c(i)\neq c(j)}} w(i,j)(1-\langle c(i),c(j)\rangle)}\\
&\quad= \frac{\sum_{(i,j)\in E\colon} w(i,j)[1-\mathbb{P}((Gc(i),Gc(j))\in (A_1\times A_1)\cup\cdots\cup(A_k\times A_k))]}{\frac{k-1}{k}\sum_{(i,j)\in E\colon c(i)\neq c(j)} w(i,j)(1-\langle c(i),c(j)\rangle)}\\
&\quad\geq \inf_{x,y\in S^{n-1}\colon \langle x,y\rangle\geq -1/(k-1)} \frac{k}{k-1}\cdot\frac{1-\mathbb{P}((Gx,Gy)\in (A_1\times A_1)\cup\cdots\cup(A_k\times A_k))}{(1-\langle x,y\rangle)}\\
&\quad= \inf_{\rho\geq -1/(k-1)} \frac{k}{k-1}\cdot\frac{1-\sum_{i=1}^k\int 1_{A_i}T_\rho 1_{A_i}d\gamma_m}{(1-\rho)}.
\end{aligned}$$

Here we used that $w(i,j)\geq 0$, and the inequality for positive numbers

$$\sum_{ij} w(i,j)a_{ij}/\sum_{ij} w(i,j)b_{ij}\geq \min_{i,j}(a_{ij}/b_{ij}).$$

3.2. **The Plurality is Stablest Conjecture.** We now briefly describe the Plurality is Stablest Conjecture. This Conjecture seems to first appear in [11]. The work [11] emphasizes the applications of this conjecture to MAX-k-CUT and to MAX-2LIN(k).

Let $n\geq 2, k\geq 3$ Let $(W_1,\ldots,W_k)$ be an orthonormal basis for the space of functions $\{g\colon\{1,\ldots,k\}\to[0,1]\}$ equipped with the inner product $\langle g,h\rangle_k := \frac{1}{k}\sum_{\sigma\in\{1,\ldots,k\}} g(\sigma)h(\sigma)$. Assume that $W_1=1$. By orthonormality, there exist $\widehat{g}(\sigma)\in\mathbb{R}$, $\sigma\in\{1,\ldots,k\}$, such that the following expression holds: $g = \sum_{\sigma\in\{1,\ldots,k\}} \widehat{g}(\sigma)W_\sigma$. Define

$$\Delta_k := \{(x_1,\ldots,x_k)\in\mathbb{R}^k \colon \forall\, 1\leq i\leq k, 0\leq x_i\leq 1, \sum_{i=1}^{k} x_i = 1\}.$$

Let $f\colon \{1,\ldots,k\}^n\to\Delta_k$, $f=(f_1,\ldots,f_k)$, $f_i\colon\{1,\ldots,k\}^n\to[0,1]$, $i\in\{1,\ldots,k\}$. Let $\sigma = (\sigma_1,\ldots,\sigma_n)\in\{1,\ldots,k\}^n$. Define $W_\sigma := \prod_{i=1}^n W_{\sigma_i}$, and let $|\sigma| := |\{i\in\{1,\ldots,n\}\colon \sigma_i\neq 1\}|$. Then there exists $\widehat{f_i}(\sigma)\in\mathbb{R}$ such that $f_i = \sum_{\sigma\in\{1,\ldots,k\}^n}\widehat{f_i}(\sigma)W_\sigma$, $i\in\{1,\ldots,k\}$.

For $\rho\in[-1,1]$ and $i\in\{1,\ldots,k\}$, define

$$T_\rho f_i := \sum_{\sigma\in\{1,\ldots,k\}^n} \rho^{|\sigma|}\widehat{f_i}(\sigma)W_\sigma, \quad T_\rho f := (T_\rho f_1,\ldots,T_\rho f_k)\in\mathbb{R}^k.$$

Let $m\geq 2$, $k\geq 3$. Let $e_j = (0,\ldots,0,1,0,\ldots,0)\in\mathbb{R}^k$ be the j^{th} unit coordinate vector. Let $\sigma\in\{1,\ldots,k\}^n$. Define $\mathrm{PLUR}_{m,k}\colon\{1,\ldots,k\}^m\to\Delta_k$ such that

$$\mathrm{PLUR}_{m,k}(\sigma) := \begin{cases} e_j & ,\text{if } |\{i\in\{1,\ldots,m\}\colon \sigma_i = j\}| > |\{i\in\{1,\ldots,m\}\colon \sigma_i = r\}|, \\ & \qquad \forall\, r\in\{1,\ldots,k\}\setminus\{j\} \\ \frac{1}{k}\sum_{i=1}^k e_i & ,\text{otherwise}\end{cases}$$

Conjecture 2 (**Plurality is Stablest Conjecture**, [10])**.** *Let $n\geq 2$, $k\geq 3$, $\rho\in[-\frac{1}{k-1},1]$, $\varepsilon>0$. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^n$. Then there exists $\tau>0$ such that, if $f\colon\{1,\ldots,k\}^n\to\Delta_k$ satisfies $\sum_{\sigma\in\{1,\ldots,k\}^n\colon\sigma_j\neq 1}(\widehat{f_i}(\sigma))^2\leq\tau$ for all $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n\}$, then*

(a) *If $\rho\in(0,1]$, and if $\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} f(\sigma) = \frac{1}{k}\sum_{i=1}^k e_i$, then*

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma),T_\rho f(\sigma)\rangle \leq \lim_{m\to\infty}\frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m}\langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle + \varepsilon.$$

(b) *If $\rho\in[-1/(k-1),0)$, then*

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma),T_\rho f(\sigma)\rangle \geq \lim_{m\to\infty}\frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m}\langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle - \varepsilon.$$

4. A Probabilistic Interpretation

We now rewrite the above conjectures using probabilistic notation and terminology.

Definition 4.1 (**Voting**)**.** Let $k\geq 2$, $n\geq 1$, $k,n\in\mathbb{Z}$. Let $f\colon\{1,\ldots,k\}^n\to\{1,\ldots,k\}$. We say that f is a *voting method* for n voters and k candidates. For $x=(x_1,\ldots,x_n)\in\{1,\ldots,k\}^n$, and for fixed $i\in\{1,\ldots,n\}$, we interpret $x_i\in\{1,\ldots,k\}$ as the vote of person i for candidate x_i. Given the votes x, the winner of the election is $f(x)\in\{1,\ldots,k\}$.

Example 4.2. From the voting perspective, the most basic functions are plurality and dictatorship. The i^{th} dictator function is defined by $f(x_1,\ldots,x_n) := x_i$.

Definition 4.3 (**Noise Stability**)**.** Let $f\colon\{1,\ldots,k\}^n\to\{1,\ldots,k\}$. Let $\rho\in(-1/(k-1),1)$. Let $\sigma=(\sigma_1,\ldots,\sigma_n),\tau=(\tau_1,\ldots,\tau_n)\in\{1,\ldots,k\}^n$ be random variables such that σ is uniformly distributed in $\{1,\ldots,k\}^n$, and for each $i\in\{1,\ldots,n\}$,

$$\mathbb{P}(\tau_i = a|\sigma_i = b) := \rho 1_{(a=b)} + (1/k)(1-\rho).$$

We define the *noise stability* of f with parameter ρ by

$$\mathbb{P}(f(\sigma) = f(\tau)).$$

Note that, for $\rho = 0$, τ and σ are independent, for $\rho = 1$, $\tau = \sigma$, and for $\rho = -1/(k-1)$, $\tau_i \neq \sigma_i$ for all $i = 1, \ldots, n$.

Definition 4.4 (**Influence**)**.** Let $f\colon \{1,\ldots,k\}^n \to \{1,\ldots,k\}$. For $i = 1,\ldots,n$, Let W_i, Z_i be iid uniform random variables on $\{1,\ldots,k\}$. For $i \in \{1,\ldots,n\}$, we define the *influence* $\mathrm{Inf}_i(f)$ of the i^{th} variable on f by

$$\mathrm{Inf}_i(f) := \mathbb{P}(f(W_1,\ldots,W_n) \neq f(W_1,\ldots,W_{i-1},Z_i,W_{i+1},\ldots,W_n)).$$

Conjecture 3 (**Plurality is Stablest Conjecture, Informal, Probabilistic**, [10])**.** *Among all voting methods where each candidate has an equal chance of winning, and every person has a small influence over the outcome of the election, the plurality function is the most noise stable.*

Conjecture 4 (**Plurality is Stablest Conjecture, Formal, Probabilistic**, [10])**.** *Let* $n \geq 2$, $k \geq 3$, $\rho \in [-\frac{1}{k-1}, 1]$, $\varepsilon > 0$. *Then there exists* $\eta > 0$ *such that, if* $f\colon \{1,\ldots,k\}^n \to \{1,\ldots,k\}$ *satisfies* $\max_{i=1,\ldots,n} \mathrm{Inf}_i f < \eta$, *then*

(a) *If* $\rho \in (0,1]$, *and if* $\mathbb{P}(f = i) = 1/k$ *for each* $i = 1,\ldots,k$, *then*

$$\mathbb{P}(f(\sigma) = f(\tau)) \leq \lim_{m\to\infty} \mathbb{P}(\mathrm{PLUR}_{m,k}(\sigma) = \mathrm{PLUR}_{m,k}(\tau)) + \varepsilon.$$

(b) *If* $\rho \in [-1/(k-1), 0)$, *then*

$$\mathbb{P}(f(\sigma) = f(\tau)) \geq \lim_{m\to\infty} \mathbb{P}(\mathrm{PLUR}_{m,k}(\sigma) = \mathrm{PLUR}_{m,k}(\tau)) - \varepsilon.$$

Recall that $\mathrm{PLUR}_{m,k}(\omega) = j$ if $|\{i \in \{1,\ldots,m\}\colon \omega_i = j\}| > |\{i \in \{1,\ldots,m\}\colon \sigma_i = r\}|$, $\forall$ $r \in \{1,\ldots,k\} \setminus \{j\}$. And the definition of $\mathrm{PLUR}_{m,k}$ for other variables does not matter for the purpose of computing the above limit.

To see the equivalence of this conjecture with Conjecture 2, let $f\colon \{1,\ldots,k\}^n \to \Delta_k$. Using the notation of Conjecture 2,

$$\begin{aligned}\mathbb{P}(f(\sigma) = f(\tau)) &= \mathbb{E}\langle f(\sigma), f(\tau)\rangle \\ &= \sum_{i=1}^{k} \mathbb{E}(f_i(\sigma) f_i(\tau)) = k^{-n} \sum_{x \in \{1,\ldots,k\}^n} \langle f(x), T_\rho f(x)\rangle. \qquad (*)\end{aligned}$$

A similar probabilistic interpretation applies to the quantity $\int 1_A e^{-tL} 1_A d\gamma_n$. In this case, let $\rho \in (0,1)$, let $e^{-t} = \rho$, and let $X = (X_1,\ldots,X_n), Y = (Y_1,\ldots,Y_n) \in \mathbb{R}^n$ be standard Gaussian random vectors such that, for all $i,j \in \{1,\ldots,n\}$, $i \neq j$, $\mathbb{E}X_iY_j = \rho 1_{(i=j)}$. Then, for any $A \subseteq \mathbb{R}^n$,

$$\int 1_A e^{-tL} 1_A d\gamma_n = \mathbb{P}((X,Y) \in A \times A).$$

Conjecture 5 (**Standard Simplex Conjecture, Probabilistic Form**)**.** *Let* $A_1,\ldots,A_k$ *be a partition of* $\mathbb{R}^n$, *and define* $B_1,\ldots,B_k$ *as in* (‡) *below.*

(a) *If* $\rho \in (0,1)$ *and if* $\gamma_n(A_i) = 1/k$ *for all* $i = 1,\ldots,k$, *then*

$$\sum_{i=1}^{k} \mathbb{P}((X,Y) \in A_i \times A_i) \leq \sum_{i=1}^{k} \mathbb{P}((X,Y) \in B_i \times B_i).$$

(b) *If* $\rho \in (-1,0)$ *(with no measure restriction on the sets* $A_1,\ldots,A_k$*), then*

$$\sum_{i=1}^{k}\mathbb{P}((X,Y)\in A_i\times A_i)\geq\sum_{i=1}^{k}\mathbb{P}((X,Y)\in B_i\times B_i).$$

To connect Conjecture 4 back to Conjecture 5, let $z_1,\ldots,z_k\in\mathbb{R}^n$ be the vertices of a regular simplex centered at the origin. Define the standard simplex partition $\{B_i\}_{i=1}^{k}$ by

$$B_i=\{x\in\mathbb{R}^n\colon \langle x,z_i\rangle=\max_{j=1,\ldots,k}\langle x,z_j\rangle\},\quad i=1,\ldots,k. \qquad (\ddagger)$$

For $\omega=(\omega_1,\ldots,\omega_n)\in\{1,\ldots,k\}^n$, define

$$V(\omega):=\sqrt{\frac{k-1}{k}}n^{-1/2}\sum_{i=1}^{n}z_{\omega_i}.$$

Then $(V(\sigma),V(\tau))$ converges weakly to (X,Y) by the Central Limit Theorem. Also, by the definition of $\mathrm{PLUR}_{m,k}$,

$$\mathrm{PLUR}_{n,k}(\omega)=j\quad\Leftrightarrow\quad[\langle V(\omega),z_i\rangle>\max_{j\neq i}\langle V(\omega),z_j\rangle]\quad\Leftrightarrow\quad V(\omega)\in B_j.$$

And therefore, using $(*)$

$$\begin{aligned}\sum_{i=1}^{k}\mathbb{P}((V(\sigma),V(\tau))\in B_i\times B_i)&=\sum_{i=1}^{k}\mathbb{P}(\mathrm{PLUR}_{n,k}(\sigma)=i,\mathrm{PLUR}_{n,k}(\tau)=i)\\&=\mathbb{P}(\mathrm{PLUR}_{n,k}(\sigma)=\mathrm{PLUR}_{n,k}(\tau)).\end{aligned}$$

Letting $n\to\infty$ and using the Central Limit theorem, this becomes

$$\sum_{i=1}^{k}\mathbb{P}((X,Y)\in B_i\times B_i)=\lim_{n\to\infty}\mathbb{P}(\mathrm{PLUR}_{n,k}(\sigma)=\mathrm{PLUR}_{n,k}(\tau)).$$

In summary, the stability of the Plurality function to the corruption of votes is exactly equal to the noise stability of the standard simplex partition $\{B_1,\ldots,B_k\}$.

To finally see Conjecture 4 as a consequence of Conjecture 5, we need to approximate a a discrete function by a Euclidean function. The main tool for this task is the invariance principle. This Theorem says that, for certain smooth multilinear polynomials g, when I swap out discrete or Gaussian inputs, the resulting distribution of g is pretty much unchanged. To show that the distributions are close to one another, we use arbitrary test functions Ψ. We first begin with a simpler form of this principle for $k=2$.

Theorem 4.5 (**Invariance Principle, Simplified Form**). [15, 10] *Let* $\{W_i\}_{i=1}^{n}$ *be iid random variables with* $\mathbb{E}W_1=0$, $\mathbb{E}W_1^2=1$, $\mathbb{E}\,|W_1|^3\leq\beta$. *Let* $g=(g_1,\ldots,g_k)$ *be a* k*-dimensional, degree* d *multilinear polynomial. So, for all* $S\subseteq\{1,\ldots,n\}$, $\exists\ \widehat{g}(S)\in\mathbb{R}^k$ *such that*

$$g(W_1,\ldots,W_n)=\sum_{S\subseteq\{1,\ldots,n\}\colon |S|\leq d}\widehat{g}(S)\prod_{i\in S}W_i.$$

Assume that for all $i\in\{1,\ldots,n\}$, $\mathbb{E}_W(g_i-\mathbb{E}_W g_i)^2\leq 1$, *and there exists* $\tau>0$ *such that, for all* $i\in\{1,\ldots,n\}$, $j\in\{1,\ldots,k\}$

$$\sum_{S\subseteq\{1,\ldots,n\}\colon i\in S}(\widehat{g_j}(S))^2=\mathrm{Inf}_i(g_j)<\tau.$$

Let $G_1, \ldots, G_k$ be iid standard Gaussians. Now, let $\Psi \colon \mathbb{R}^k \to \mathbb{R}$ be a C^3 function such that $\left|\Psi^{(r)}\right| \leq C$ for all multi-indices $r = (r_1, r_2, r_3) \in \{1, \ldots, k\}^3$. Then

$$|\mathbb{E}\Psi(g(W_1, \ldots, W_k)) - \mathbb{E}\Psi(f(G_1, \ldots, G_k))| \leq Cd\beta^{1/3}\tau^{1/(8d)} = O(\sqrt{\tau}).$$

The invariance principle has a few stronger forms. We state one below.

Let $\{W_j\}_{j=1}^k$ be a basis for the space of functions $\{1, \ldots, k\} \to \mathbb{R}$ so that this basis is orthonormal with respect to the inner product $\langle h, s\rangle = k^{-1}\sum_{j=1}^k h(j)s(j)$. Assume also that $W_1 = 1$. For each $i \in \{1, \ldots, n\}$, let $W_{i,1} = 1$, and let $\{W_{i,j}\}_{j=2}^k$ be an independent copy of $\{W_j\}_{j=2}^k$. Then the set of functions $\{\prod_{i=1}^n W_{i,\sigma_i}\}_{\sigma\in\{1,\ldots,k\}^n}$ forms an orthonormal basis for the space of functions $\{1, \ldots, k\}^n \to \mathbb{R}$. Let $G_1 = 1$, and let $G_2, \ldots, G_k$ be iid standard Gaussians, and for each $i \in \{1, \ldots, n\}$, let $\{G_{i,j}\}_{j=2}^k$ be an independent copy of $\{G_j\}_{j=2}^k$. Also, for each $i \in \{1, \ldots, n\}$, let $G_{i,1} = 1$.

Theorem 4.6 (**Invariance Principle, Stronger Form**). [15, 10] *Let $g = (g_1, \ldots, g_k)$ be a k-dimensional, degree d multilinear polynomial. So, for all $\sigma \subseteq \{1, \ldots, k\}^n$, $\exists\ \widehat{g}(\sigma) \in \mathbb{R}^k$ such that*

$$g(\{W_{i,j}\}_{1\leq i\leq n, 1\leq j\leq k}) = \sum_{\substack{\sigma\in\{1,\ldots,k\}^n\colon \\ |\{j\in\{1,\ldots,n\}\colon \sigma_j\neq 1\}|\leq d}} \widehat{g}(\sigma)\prod_{i=1}^n W_{i,\sigma_i}(x_i).$$

Assume that for all $i \in \{1, \ldots, n\}$, $\mathbb{E}_W(g_i - \mathbb{E}_W g_i)^2 \leq 1$, and there exists $\tau > 0$ such that, for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, k\}$

$$\sum_{\sigma\in\{1,\ldots,k\}^n\colon \sigma_i\neq 1} (\widehat{g_j}(\sigma))^2 = \mathrm{Inf}_i(g_j) < \tau.$$

Now, let $\Psi \colon \mathbb{R}^k \to \mathbb{R}$ be a C^3 function such that $\left|\Psi^{(r)}\right| \leq C$ for all multi-indices $r = (r_1, r_2, r_3) \in \{1, \ldots, k\}^3$. Then

$$\left|\mathbb{E}\Psi(g(\{W_{i,j}\}_{\substack{1\leq i\leq n,\\ 1\leq j\leq k}})) - \mathbb{E}\Psi(g(\{G_{i,j}\}_{\substack{1\leq i\leq n,\\ 1\leq j\leq k}}))\right| \leq 2dCk^3(8/k)^d\sqrt{\tau} = O(\sqrt{\tau}).$$

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