STANDARD SIMPLICES AND PLURALITIES FOR UNEQUAL MEASURES ARE NOT THE MOST NOISE STABLE

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Abstract. We consider a natural generalization of Borell's isoperimetric inequality for Gaussian space for multiple sets of unequal measures. We show that this conjecture is false. The argument combines the first variation with a simple analyticity argument, the latter being a new ingredient. (joint work with Elchanan Mossel and Joe Neeman [7])

1. Introduction

The general motivation for investigating Gaussian isoperimetric problems comes from two connections between discrete domains and continuous domains: the Invariance Principle and Semidefinite Programming. These concepts are illustrated in the following diagram.

Analysis on the discrete space $\{1,2,\ldots,k\}^n$	Invariance Principle, [3, 11] $\Longleftrightarrow$	Analysis on $(\mathbb{R}^n, e^{-\|x\|_2^2/2})$
(Optimal) Algorithms for variables with k values	Semidefinite Programming, [9, 8] $\Longleftrightarrow$	(Optimal) Partitions of $\mathbb{R}^n$ into k sets

Since the discrete and continuous domains are connected in this way, working in one realm implies results in the other. We will therefore focus on the lower right corner of this diagram. For simplicity, we begin by discussing the case $k = 2$.

2. Borell's Theorem for Noise Stability

Theorem 2.1 (**Borell's Theorem, Informal**, [1]). *The most noise-stable Euclidean sets are the half spaces.*

We will investigate variants of Theorem 2.1 for multiple sets. We first establish some notation to make Theorem 2.1 a formal statement.

Definition 2.2. Let $z \in \mathbb{R}^n$, $t \in \mathbb{R}$. Define $H := \{x \in \mathbb{R}^n : \langle x, z\rangle \geq t\}$. We call $H \subseteq \mathbb{R}^n$ a **half space**.

Let $n \geq 1$, $n \in \mathbb{Z}$, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let dx denote Lebesgue measure on $\mathbb{R}^n$, and define $\gamma_n(x) := e^{-(x_1^2+\cdots+x_n^2)/2}(2\pi)^{-n/2}$.

Definition 2.3 (**Ornstein-Uhlenbeck operator**). Let $f : \mathbb{R}^n \to [0,1]$. For $x \in \mathbb{R}^n$ and $\rho \in (-1,1)$, define

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) = \mathbb{E}f(x\rho + Y\sqrt{1-\rho^2}), \quad Y \sim N(0,1).$$

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Definition 2.4 (**Noise Stability**)**.** Let $A \subseteq \mathbb{R}^n$. Define the **Noise Stability** of A with parameter ρ as

$$\int_{\mathbb{R}^n} 1_A \cdot T_\rho 1_A d\gamma_n.$$

Theorem 2.5 (**Borell's Theorem, Formal**, [1, 2])**.** *Let $A \subseteq \mathbb{R}^n$ and let $B := \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Let $y \in \mathbb{R}^n$ such that $\gamma_n(A) = \gamma_n(B + y)$. Then, for all $\rho \in (0, 1)$*

$$\int_{\mathbb{R}^n} 1_A \cdot T_\rho 1_A d\gamma_n \leq \int_{\mathbb{R}^n} 1_{(B+y)} \cdot T_\rho 1_{(B+y)} d\gamma_n.$$

Also, for all $\rho \in (-1, 0)$,

$$\int_{\mathbb{R}^n} 1_A \cdot T_\rho 1_A d\gamma_n \geq \int_{\mathbb{R}^n} 1_{(B+y)} \cdot T_\rho 1_{(B+y)} d\gamma_n.$$

For modern proofs with additional stability statements, see [10, 5]. Intuitively, since the Gaussian measure and the Ornstein-Uhlenbeck operator are highly symmetric, we expect that the set optimizing noise stability is also highly symmetric. So, a posteriori, the inequality of Theorem 2.5 seems sensible.

Theorem 2.5 implies the Majority is Stablest Theorem [11, Theorem 4.4]. This Theorem says that the function $f : \{-1, 1\}^n \to \{-1, 1\}$ of fixed expected value $2^{-n} \sum_{\sigma \in \{-1,1\}} f(\sigma)$ and small derivatives which is most stable to noise is a majority function. A majority function is a function of the form $\text{sign}(\sigma_1 + \cdots + \sigma_n + c)$, where $(\sigma_1, \ldots, \sigma_n) \in \{0, 1\}^n$ and $c \in \mathbb{R}$. Here we interpret a function $f : \{-1, 1\}^n \to \{-1, 1\}$ as a voting method, where the domain corresponds to n voters who vote for two candidates, and given the votes $(\sigma_1, \ldots, \sigma_n) \in \{-1, 1\}^n$, the value of the function $f(\sigma_1, \ldots, \sigma_n)$ is the winner of the election.

Theorem 2.5 also implies that the Goemans-Williams semidefinite program which approximately solves the MAX-CUT is the best possible, assuming the Unique Games Conjecture [9, Theorem 1].

From the voting interpretation or from applications to the Unique Games Conjecture [9], it is natural to look for analogues of Theorem 2.5 for multiple sets. We will not mention the case of two sets, since the noise sensitivity of two sets which partition $\mathbb{R}^n$ is equivalently described by the noise sensitivity of one these sets. We will now state such a Conjecture for multiple, and we will then discuss why this conjecture is not true. For simplicity, we will concentrate on the case of $k = 3$ sets in $\mathbb{R}^2$.

Definition 2.6. Let $A_1, A_2, A_3 \subseteq \mathbb{R}^2$ be measurable. We say that $\{A_i\}_{i=1}^3$ is a **partition** of $\mathbb{R}^2$ if $\cup_{i=1}^3 A_i = \mathbb{R}^2$, and $\gamma_2(A_i \cap A_j) = 0$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$.

Conjecture 1 (**Shifted Simplex Conjecture**)**.** *Let $\rho \in (-1, 1)$. Let $\{A_i\}_{i=1}^3$ be a partition of $\mathbb{R}^2$. Let $0 < a_1, a_2, a_3 < 1$ with $\sum_{i=1}^3 a_i = 1$. Assume that $\gamma_2(A_i) = a_i$ for all $i = 1, 2, 3$, and assume that*

$$(a_1, a_2, a_3) \neq (1/3, 1/3, 1/3).$$

Then there exist $B_1, B_2, B_3 \subseteq \mathbb{R}^2$ three disjoint nonempty sectors centered at the origin and there exists $y \in \mathbb{R}^2$ such that the following statements hold. We have $\gamma_2(B_i + y) = a_i$, for all $i = 1, 2, 3$, and

(a) *If* $\rho \in (0,1]$*, then*

$$\sum_{i=1}^{3}\int_{\mathbb{R}^2} 1_{A_i}\cdot T_\rho 1_{A_i} d\gamma_2 \le \sum_{i=1}^{3}\int_{\mathbb{R}^2} 1_{(B_i+y)}\cdot T_\rho 1_{(B_i+y)} d\gamma_2. \tag{1}$$

(b) *If* $\rho \in [-1,0)$*, then*

$$\sum_{i=1}^{3}\int_{\mathbb{R}^2} 1_{A_i}\cdot T_\rho 1_{A_i} d\gamma_2 \ge \sum_{i=1}^{3}\int_{\mathbb{R}^2} 1_{(B_i+y)}\cdot T_\rho 1_{(B_i+y)} d\gamma_2. \tag{2}$$

Remark 2.7. Letting $\rho \to 1^-$, Conjecture 1 becomes an isoperimetric problem that is known to hold for (a_1, a_2, a_3) in a neighborhood of $(1/3, 1/3, 1/3)$, where $a_1 + a_2 + a_3 = 1$ [4]. As $\rho \to 1^-$, the noise stability converges to Gaussian perimeter, when normalized appropriately. Also, Conjecture 1 for $(a_1, a_2, a_3) = (1/3, 1/3, 1/3)$ is known as the Standard Simplex Conjecture (for three sets), and it is known to hold for all $0 < \rho < \rho_0(n)$, for some $\rho_0(n)$ [6]. Therefore, the following negative result may come as a surprise.

Theorem 2.8 (**Main Result**)**.** [7, Theorem 2.6] *Conjecture 1 does not hold.*

As a Corollary, if the Plurality is Stablest Conjecture is true, then it is barely true.

Corollary 2.9. *Conjecture 4 below does not hold.*

Remark 2.10. Theorem 2.8 and Corollary 2.9 have analogous statements for more than three sets [7].

3. A Sketch of the Proof of Theorem 2.8

Theorem 2.8 follows from a few assertions. We first establish some notation and exploit some symmetry. For simplicity, let $\rho \in (0,1)$ and assume that $B_1 \cap B_2 \subseteq \{(x,y) \in \mathbb{R}^2 \colon x = 0\}$. Write $y = (y_1, y_2) \in \mathbb{R}^2$. Let L be the infinite line such that $L \supseteq (B_1 + y) \cap (B_2 + y)$. Parametrize L so that $L = \{(y_1, ty_2) \colon t \in \mathbb{R}\}$. Without loss of generality, $\exists\, T > 0$ such that $(B_1 + y) \cap (B_2 + y) \supseteq \{(y_1, ty_2) \colon t > T\}$.

From a first variation argument [7, Lemma 3.1], it suffices to show that $T_\rho(1_{B_1+y} - 1_{B_2+y})(x)$ is not constant for $x \in (B_1 + y) \cap (B_2 + y)$. This property of being nonconstant then follows from the following itemized assertions [7, Lemma 4.1]. Let $t \in \mathbb{R}$ and define

$$f(t) := T_\rho(1_{(B_1+y)} - 1_{(B_2+y)})(y_1, ty_2).$$

Then the following three properties hold (which we will not prove).

- $|\lim_{t\to\infty} f(t)| = 2\gamma_1[0, |y_1|\, c(\rho)]$, for some nonzero constant $c(\rho)$.
- $\lim_{t\to-\infty} f(t) = 0$.
- $f(t)$ is a holomorphic function of t.

We conclude that f is constant on some line segment in L if and only if $y_1 = 0$. For $i, j \in \{1,2,3\}$ with $i \neq j$, define $f_{ij}(t) := T_\rho(1_{B_i+y} - 1_{B_j+y})(y_1, ty_2)$. Applying the previous observation to any two-element subset $\{i,j\} \subseteq \{1,2,3\}$ shows: f_{ij} is constant for all $i, j \in \{1,2,3\}$, $i \neq j$ if and only if $y = 0$. If $y = 0$, then each $B_i + y = B_i$, $i = 1,2,3$ is a dilation invariant cone with $0 \in B_i$. Recall that $\lim_{t\to-\infty} f_{ij}(t) = 0$ for all $i, j \in \{1,2,3\}$, $i \neq j$. So, if f_{ij} is constant for all $i, j \in \{1,2,3\}$, $i \neq j$, then $f_{ij} = 0$ for all $i, j \in \{1,2,3\}$. That

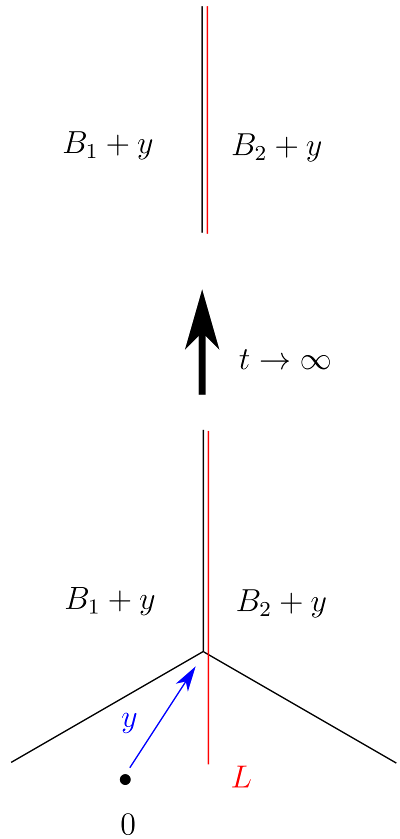

FIGURE 1. Proof of Theorem 2.8

is, $\gamma_2(B_i) = T_\rho 1_{B_i}(0) = T_\rho 1_{B_j}(0) = \gamma_2(B_j)$, for all $i, j \in \{1, 2, 3\}$, $i \neq j$. In conclusion, $\gamma_2(B_i) = 1/3$ for all $i = 1, 2, 3$. This concludes Theorem 2.8, since Conjecture 1 assumes that $(a_1, a_2, a_3) \neq (1/3, 1/3, 1/3)$.

4. The Plurality is Stablest Conjecture for Three Voters

Definition 4.1 (**Voting**). Let $f : \{1, 2, 3\}^n \to \{1, 2, 3\}$. We say that f is a *voting method* for n voters and 3 candidates. For $x = (x_1, \ldots, x_n) \in \{1, 2, 3\}^n$, and for fixed $i \in \{1, \ldots, n\}$, we interpret $x_i \in \{1, 2, 3\}$ as the vote of person i for candidate x_i. Given the votes x, the winner of the election is $f(x) \in \{1, 2, 3\}$.

Example 4.2. From the voting perspective, the most basic functions are plurality and dictatorship. The i^{th} dictator function is defined by $f(x_1, \ldots, x_n) := x_i$.

Definition 4.3 (**Noise Stability**). Let $f : \{1, 2, 3\}^n \to \{1, 2, 3\}$. Let $\rho \in (-1/2, 1)$. Let $\sigma = (\sigma_1, \ldots, \sigma_n), \tau = (\tau_1, \ldots, \tau_n) \in \{1, 2, 3\}^n$ be random variables such that σ is uniformly distributed in $\{1, 2, 3\}^n$, and for each $i \in \{1, \ldots, n\}$,

$$\mathbb{P}(\tau_i = a | \sigma_i = b) := \rho 1_{(a=b)} + (1/3)(1 - \rho).$$

We define the *noise stability* of f with parameter ρ by

$$\mathbb{P}(f(\sigma) = f(\tau)).$$

Note that, for $\rho = 0$, τ and σ are independent, for $\rho = 1$, $\tau = \sigma$, and for $\rho = -1/2$, $\tau_i \neq \sigma_i$ for all $i = 1, \ldots, n$.

Definition 4.4 (**Influence**)**.** Let $f\colon \{1,2,3\}^n \to \{1,2,3\}$. For $i = 1, \ldots, n$, Let W_i, Z_i be iid uniform random variables on $\{1,2,3\}$. For $i \in \{1, \ldots, n\}$, we define the *influence* $\mathrm{Inf}_i(f)$ of the i^{th} variable on f by

$$\mathrm{Inf}_i(f) := \mathbb{P}(f(W_1, \ldots, W_n) \neq f(W_1, \ldots, W_{i-1}, Z_i, W_{i+1}, \ldots, W_n)).$$

Conjecture 2 (**Plurality is Stablest Conjecture, Informal**, [8])**.** *Among all voting methods where each candidate has an equal chance of winning, and every person has a small influence over the outcome of the election, the plurality function is the most noise stable.*

Conjecture 3 (**Plurality is Stablest Conjecture, Formal**, [8])**.** *Let $n \geq 2$, $\rho \in [-1/2, 1]$, $\varepsilon > 0$. Then $\exists\ \eta > 0$ such that, if $f\colon \{1,2,3\}^n \to \{1,2,3\}$ satisfies $\max_{i=1,\ldots,n} \mathrm{Inf}_i f < \eta$, then*

(a) *If $\rho \in (0, 1]$, and if $\mathbb{P}(f = i) = 1/k$ for each $i = 1, \ldots, k$, then*

$$\mathbb{P}(f(\sigma) = f(\tau)) \leq \lim_{m\to\infty} \mathbb{P}(\mathrm{PLUR}_{m,3}(\sigma) = \mathrm{PLUR}_{m,3}(\tau)) + \varepsilon.$$

(b) *If $\rho \in [-1/2, 0)$, then*

$$\mathbb{P}(f(\sigma) = f(\tau)) \geq \lim_{m\to\infty} \mathbb{P}(\mathrm{PLUR}_{m,3}(\sigma) = \mathrm{PLUR}_{m,3}(\tau)) - \varepsilon.$$

Recall that $\mathrm{PLUR}_{m,3}(\omega) := j$ if $|\{i \in \{1, \ldots, m\}\colon \omega_i = j\}| > |\{i \in \{1, \ldots, m\}\colon \sigma_i = r\}|$, $\forall$ $r \in \{1,2,3\} \setminus \{j\}$. And the definition of $\mathrm{PLUR}_{m,3}$ for other variables does not matter for the purpose of computing the above limit.

We now state a variant of the Plurality is Stablest Conjecture that ends up being false.

Conjecture 4 (**Shifted Plurality is Stablest Conjecture**, [8])**.** *Let $n \geq 2$, $\rho \in (0, 1)$, $\varepsilon > 0$. Let $(\alpha, \beta) \neq (0, 0)$, $\alpha, \beta \in \mathbb{R}$. Let $\rho > 0$. Let $n \geq 2$, and let $\widetilde{Q}$ be the probability measure on $\{1,2,3\}$ given by*

$$\widetilde{Q}(1) = \frac{1}{3} + \alpha n^{-1/2}, \quad \widetilde{Q}(2) = \frac{1}{3} + \beta n^{-1/2}, \quad \widetilde{Q}(3) = \frac{1}{3} - (\alpha + \beta) n^{-1/2}.$$

Let $\widetilde{P}$ be the distribution on $\{1,2,3\}^2$ such that, for each $x, y \in \{1,2,3\}$,

$$\widetilde{P}(x, y) = \rho 1_{(x=y)} \widetilde{Q}(x) + (1 - \rho) \widetilde{Q}(x) \widetilde{Q}(y).$$

Let $\sigma, \tau \in \{1,2,3\}^n$ so that (σ, τ) is distributed according to $\mathbb{P} := \widetilde{P}^n$. Let $f_n\colon \{1,2,3\}^n \to \{1,2,3\}$ be a sequence of functions such that $\lim_{n\to\infty} \max_{i=1,\ldots,n} \mathrm{Inf}_i f = 0$. Then

$$\lim_{n\to\infty} \mathbb{P}(f_n(\sigma) = f_n(\tau)) \leq \lim_{m\to\infty} \mathbb{P}(\mathrm{PLUR}_{m,3}(\sigma) = \mathrm{PLUR}_{m,3}(\tau)).$$

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