

# NONCOMMUTATIVE MAJORIZATION PRINCIPLES AND GROTHENDIECK'S INEQUALITY

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ABSTRACT. The seminal invariance principle of Mossel-O'Donnell-Oleszkiewicz implies the following. Suppose we have a multilinear polynomial  $Q$ , all of whose partial derivatives are small. Then the distribution of  $Q$  on i.i.d. uniform  $\{-1, 1\}$  inputs is close to the distribution of  $Q$  on i.i.d. standard Gaussian inputs. The case that  $Q$  is a linear function recovers the Berry-Esséen Central Limit Theorem. In this way, the invariance principle is a nonlinear version of the Central Limit Theorem. We prove the following version of one of the two inequalities of the invariance principle, which we call a majorization principle. Suppose we have a multilinear polynomial  $Q$  with matrix coefficients, all of whose partial derivatives are small. Then, for any even  $K > 1$ , the  $K^{\text{th}}$  moment of  $Q$  on i.i.d. uniform  $\{-1, 1\}$  inputs is larger than the  $K^{\text{th}}$  moment of  $Q$  on (carefully chosen) random matrix inputs, minus a small number. The exact statement must be phrased carefully in order to avoid being false. Time permitting, we discuss applications of this result to anti-concentration, and to computational hardness for the noncommutative Grothendieck inequality. (joint with Thomas Vidick)

## 1. BERRY-ESSÉEN CENTRAL LIMIT THEOREM

Let  $n \in \mathbb{N}$  be a positive integer. Let  $x_1, \dots, x_n$  be commutative indeterminate variables, and let

$$Q(x_1, \dots, x_n) = \frac{x_1 + \dots + x_n}{\sqrt{n}}.$$

Let  $b_1, \dots, b_n$  be i.i.d. uniform random variables in  $\{-1, 1\}$ , and let  $g_1, \dots, g_n$  be i.i.d. standard Gaussian random variables. Note that  $\mathbb{E}b_1 = \mathbb{E}g_1 = 0$ , and  $\mathbb{E}b_1^2 = \mathbb{E}g_1^2 = 1$ , so that the first and second moments of  $b_1$  and  $g_1$  are identical. (Also,  $\mathbb{E}|b_1|^3$  and  $\mathbb{E}|g_1|^3$  are finite.) Let

$$\|Q\|_2 := (\mathbb{E}|Q(b_1, \dots, b_n)|^2)^{1/2}.$$

Note that  $\|Q\|_2 = 1$  and  $\|Q\|_2 := (\mathbb{E}|Q(g_1, \dots, g_n)|^2)^{1/2}$ .

We then have the following quantitative version of the Central Limit Theorem.

**Theorem 1.1 (Berry-Esséen Central Limit Theorem).**

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(Q(b_1, \dots, b_n) \leq t) - \mathbb{P}(Q(g_1, \dots, g_n) \leq t)| \leq 3 \max_{i=1, \dots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2.$$

If this expression looks unfamiliar, note that  $Q(g_1, \dots, g_n)$  has a standard Gaussian distribution, and  $\left\| \frac{\partial}{\partial x_i} Q \right\|_2 = 1/\sqrt{n}$  for all  $i \in \{1, \dots, n\}$ .

Theorem 1.1 can also be extended to moments of  $Q$ . For example,

**Theorem 1.2 (Berry-Esséen Central Limit Theorem).**

$$|\mathbb{E} |Q(b_1, \dots, b_n)|^4 - \mathbb{E} |Q(g_1, \dots, g_n)|^4| \leq 240 \max_{i=1, \dots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2.$$

**Remark 1.3.** A similar statement can be made for higher moments of  $Q$ .

## 2. THE INVARIANCE PRINCIPLE

The commutative invariance principle [Rot79, Cha06, MOO10] particularly implies that Theorem 1.2 holds for multilinear polynomials.

**Theorem 2.1 (Invariance Principle for Fourth Moments, [MOO10]).** *Let  $d \in \mathbb{N}$ . Let  $Q(x_1, \dots, x_n)$  be a multilinear polynomial of degree  $d$ , so that*

$$Q(x_1, \dots, x_n) = \sum_{S \subseteq \{1, \dots, n\}: |S| \leq d} c_S \prod_{i \in S} x_i, \quad c_S \in \mathbb{R}, \forall S \subseteq \{1, \dots, n\}.$$

*Assume that  $\|Q\|_2 \leq 1$ . Then*

$$|\mathbb{E} |Q(b_1, \dots, b_n)|^4 - \mathbb{E} |Q(g_1, \dots, g_n)|^4| \leq 24 \cdot 10^d \max_{i=1, \dots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2.$$

**Remark 2.2.** The paper [MOO10] has been very influential, with over 300 citations. Part of the interest in Theorem 2.1 is its use for proving hardness for computational problems. For example, Theorem 2.1 is used to prove sharp computational hardness for the MAX-CUT problem, assuming the Unique Games Conjecture [KKMO07]. The MAX-CUT problem asks for the partition of the vertices of an undirected graph into two sets  $S$  and  $S^c$  that maximizes the number of edges that go from  $S$  to  $S^c$ . This problem is  $NP$ -hard, but we know how to find a cut of a graph which cuts about .878567 times the maximum possible number of cut edges in polynomial time. This number .878567 is the best possible approximation we can get in polynomial time, assuming the Unique Games Conjecture.

**Remark 2.3.** Note that, other than the term  $\max_{i=1, \dots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2$ , the upper bound in Theorem 2.1 does not depend on  $n$ .

Note that the assumption  $\|Q\|_2 \leq 1$  allows the statement of Theorem 2.1 itself to not be homogeneous. That is, replacing  $Q$  by  $\lambda Q$  for some  $\lambda > 0$  scales like  $\lambda^4$  on the left side of the Theorem, but like  $\lambda$  on the right side. But if  $\lambda > 1$ , the assumption of the Theorem no longer holds.

One of the main technical tools used in Theorem 2.1 is the hypercontractive inequality.

**Theorem 2.4 ((4, 2) Hypercontractive Inequality, [Bon70, Nel73, Gro75]).** *Let  $Q$  be a multilinear polynomial of degree  $d \in \mathbb{N}$ . Then*

$$\mathbb{E} |Q(b_1, \dots, b_n)|^4 \leq 9^d [\mathbb{E} |Q(b_1, \dots, b_n)|^2]^2.$$

$$\mathbb{E} |Q(g_1, \dots, g_n)|^4 \leq 9^d [\mathbb{E} |Q(g_1, \dots, g_n)|^2]^2.$$

*Very rough proof sketch of Theorem 2.1.* Theorem 2.1 is proven using the Lindeberg replacement method. That is, we write

$$\begin{aligned} & \left| \mathbb{E} |Q(b_1, \dots, b_n)|^4 - \mathbb{E} |Q(g_1, \dots, g_n)|^4 \right| \\ &= \left| \sum_{i=1}^n \mathbb{E} |Q(b_1, \dots, b_i, g_{i+1}, \dots, g_n)|^4 - \mathbb{E} |Q(b_1, \dots, b_{i-1}, g_i, \dots, g_n)|^4 \right| \\ &\leq \sum_{i=1}^n \left| \mathbb{E} |Q(b_1, \dots, b_i, g_{i+1}, \dots, g_n)|^4 - \mathbb{E} |Q(b_1, \dots, b_{i-1}, g_i, \dots, g_n)|^4 \right|. \end{aligned}$$

And each of the final terms is bounded separately. For example, if  $Q(x_1, x_2) = (x_1 + x_2)/\sqrt{2}$ , we need to bound

$$\mathbb{E}(b_1 + b_2)^4 - \mathbb{E}(b_1 + g_2)^4.$$

Multiplying out all terms and using  $\mathbb{E}b_1 = \mathbb{E}b_1^3 = \mathbb{E}g_1 = \mathbb{E}g_1^3 = 0$ , we get

$$\begin{aligned} \left| \mathbb{E}(b_1 + b_2)^4 - \mathbb{E}(b_1 + g_2)^4 \right| &= \left| 6\mathbb{E}b_1^2\mathbb{E}b_2^2 - 6\mathbb{E}b_1^2\mathbb{E}g_2^2 + \mathbb{E}b_2^4 - \mathbb{E}g_2^4 \right| \\ &= \left| \mathbb{E}b_2^4 - \mathbb{E}g_2^4 \right|, \quad \text{using } \mathbb{E}b_2^2 = \mathbb{E}g_2^2 \\ &\leq \left| \mathbb{E}b_2^4 \right| + \left| \mathbb{E}g_2^4 \right|. \end{aligned}$$

We then bound the final terms using the hypercontractive inequality. These final terms are bounded by the right hand side of Theorem 2.1. So,

$$\begin{aligned} \left| \mathbb{E} |Q(b_1, b_2)|^4 - \mathbb{E} |Q(g_1, g_2)|^4 \right| &\leq \frac{2}{4} 9^d (|\mathbb{E}b_2^2|^2 + |\mathbb{E}g_2^2|^2) \leq 2 \cdot 9^d \sum_{i=1}^2 \left\| \frac{\partial}{\partial x_i} Q \right\|_2^4 \\ &\leq 2 \cdot 9^d \max_{j=1,2} \left\| \frac{\partial}{\partial x_j} Q \right\|_2^2 \cdot \sum_{i=1}^2 \left\| \frac{\partial}{\partial x_i} Q \right\|_2^2 \leq 2d \cdot 9^d \cdot \max_{j=1,2} \left\| \frac{\partial}{\partial x_j} Q \right\|_2^2. \end{aligned}$$

□

**Remark 2.5.** To prove the invariance principle, Theorem 2.1, we really only needed: the first and second moments of the variables match ( $\mathbb{E}b_1 = \mathbb{E}g_1$ ,  $\mathbb{E}b_1^2 = \mathbb{E}g_1^2$ ), and (4, 2) hypercontractivity holds for both variables (Theorem 2.4). These are the only assumptions needed to prove the invariance principle.

**Remark 2.6.** The Lindeberg replacement method is somewhat analogous to a martingale differencing scheme, except in a typical martingale differencing, errors are controlled in the  $L_\infty$  norm. However, here the errors are being controlled in an  $L_2$  norm, which is typically more difficult to control.

### 3. NONCOMMUTATIVE MAJORIZATION PRINCIPLE

**Remark 3.1.** We would like a noncommutative version of Theorem 2.1. Our proof of Theorem 2.1 uses the Lindeberg replacement method, though the details are a bit different than before. Namely, variables no longer commute! Also, we require a version of hypercontractivity for noncommutative variables

Let  $m \in \mathbb{N}$ . Let  $A$  be an  $m \times m$  matrix. Define  $|A| = (AA^*)^{1/2}$ .

Let  $G_1, \dots, G_n$  be i.i.d. random  $m \times m$  matrices.

**Definition 3.2 (Noncommutative Multilinear Polynomial).** Let  $X_1, \dots, X_n$  be noncommutative  $m \times m$  indeterminate variables. Let  $Q(X_1, \dots, X_n)$  be a noncommutative multilinear polynomial of degree  $d \in \mathbb{N}$ . That is, for any  $S \subseteq \{1, \dots, n\}$ , there exists  $m \times m$  matrices  $c_S$  such that

$$Q(X_1, \dots, X_n) = \sum_{S \subseteq \{1, \dots, n\}: |S| \leq d} c_S \prod_{i \in S} X_i.$$

Also, the product terms are in increasing order, so e.g.  $\prod_{i=1,2,3} X_i = X_1 X_2 X_3$ .

In order to state the majorization principle, we need to extend  $Q$  to  $p \times p$  input matrices, where  $p > m$ .

**Definition 3.3 (High-Dimensional Embedding of  $Q$ ).** Let  $X_1, \dots, X_n$  be noncommutative  $p \times p$  indeterminate variables. Define

$$Q^\iota(X_1, \dots, X_n) = \sum_{S \subseteq \{1, \dots, n\}: |S| \leq d} \begin{pmatrix} c_S & 0 \\ 0 & 0 \end{pmatrix} \prod_{i \in S} X_i$$

**Theorem 3.4 ((4, 2) Noncommutative Hypercontractive Inequality).** [Gro72] *Let  $Q$  be a noncommutative multilinear polynomial of degree  $d$ . Then*

$$\mathbb{E} \text{Tr} |Q(b_1, \dots, b_n)|^4 \leq 9^d [\mathbb{E} \text{Tr} |Q(b_1, \dots, b_n)|^2]^2.$$

When we use  $b_1, \dots, b_n$  as inputs into  $Q$ , we use the convention that  $b_1, \dots, b_n$  denotes  $b_1 I, \dots, b_n I$ , where  $I$  is the identity matrix.

Define

$$\|Q\|_2 := (\mathbb{E} \text{Tr} |Q(b_1, \dots, b_n)|^2)^{1/2}.$$

**Theorem 3.5 (Noncommutative Majorization Principle for Fourth Moments).** *Let  $p > m$ ,  $p, m \in \mathbb{N}$ . Let  $d \in \mathbb{N}$ . Let  $Q(X_1, \dots, X_n)$  be a noncommutative multilinear polynomial of degree  $d$ . Assume that  $\|Q\|_2 \leq \sqrt{m}$ . Let  $H_1, \dots, H_n$  be i.i.d. uniformly random  $p \times p$  unitary matrices. Let  $G_1, \dots, G_n$  be i.i.d. random  $m \times m$  matrices which satisfy  $\mathbb{E} G_1 G_1^* = I$ ,  $\mathbb{E} G_1 = 0$ , and  $\|\mathbb{E} |G_1|^4\| \leq 10$ . Then*

$$\begin{aligned} & \mathbb{E} \frac{1}{m} \text{Tr} \left| Q^\iota \left( \begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} H_1, \dots, \begin{pmatrix} G_n & 0 \\ 0 & 0 \end{pmatrix} H_n \right) \right|^4 \\ & \leq \mathbb{E} \frac{1}{m} \text{Tr} |Q(b_1, \dots, b_n)|^4 + m^4 2^{40d} \max_{i=1, \dots, n} \left\| \frac{\partial}{\partial x_i} Q \right\|_2 + O_{m,n} \left( \frac{1}{p} \right). \end{aligned}$$

**Remark 3.6.** If  $i \in \{1, \dots, n\}$ , then  $\frac{\partial}{\partial x_i} Q(X_1, \dots, X_n) = \sum_{S \subseteq \{1, \dots, n\}: i \in S} c_S \prod_{i \in S} X_i$ .

**Remark 3.7.** In our application, we use random matrices  $G_1, \dots, G_m$  constructed as follows. Let  $U_1, \dots, U_N$  be  $m \times m$  matrices such that  $\sum_{i=1}^N U_i U_i^* = I$ . Let  $g_1, \dots, g_N$  be i.i.d. standard complex Gaussian random variables. Let  $G_1 = \sum_{i=1}^N g_i U_i$ , and let  $G_2, \dots, G_m$  be i.i.d. copies of  $G_1$ . Then  $G_1, \dots, G_m$  are (6, 2) hypercontractive,  $\mathbb{E} G_1 = 0$  and  $\mathbb{E} G_1 G_1^* = I$ .

**Remark 3.8.** We use Theorem 3.5 to prove a computational hardness result for the noncommutative Grothendieck inequality.

**Theorem 3.9 (Grothendieck’s Inequality, [Gro53, LP68, AN06, BMMN13]).** *There exists  $1 \leq K_G < \frac{\pi}{2 \log(1+\sqrt{2})}$  such that the following holds. Let  $n \in \mathbb{N}$  and let  $a_{ij}$  be an  $n \times n$  real matrix. Then*

$$\max_{\substack{x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^{2n-1} \\ \langle x_i, x_i \rangle = \langle y_i, y_i \rangle = 1 \forall i \in \{1, \dots, n\}}} \sum_{i,j=1}^n a_{ij} \langle x_i, y_j \rangle \leq K_G \max_{\varepsilon_1, \dots, \varepsilon_n, \delta_1, \dots, \delta_n \in \{\pm 1\}} \sum_{i,j=1}^n a_{ij} \varepsilon_i \delta_j. \quad (1)$$

**Theorem 3.10 (Noncommutative Grothendieck Inequality, Complex Case, [Pis78, Haa85, NRV14]).** *Let  $n \in \mathbb{N}$  and let  $M_{ijkl} \in M_n(M_n(\mathbb{C}))$ . Then*

$$\sup_{U, V \in \mathcal{U}_n(\mathbb{C}^N)} \sum_{i,j,k,\ell=1}^n M_{ijkl} \langle U_{ij}, V_{k\ell} \rangle \leq 2 \cdot \sup_{X, Y \in \mathcal{U}_n} \sum_{i,j,k,\ell=1}^n M_{ijkl} X_{ij} Y_{k\ell}. \quad (2)$$

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