Problem Set 6

Recall that a subset $M \subset \mathbb{R}^n$ is called a k-dimensional C^l-manifold (a k-dim. C^l-MF) if every $p \in M$ has an open neighborhood $U \ni p$ and a diffeomorphism $\Psi \in C^l(U;\mathbb{R}^n)$ from U onto $V := \Psi(U)$, such that

$$\Psi(U \cap M) \;=\; V \cap (\mathbb{R}^k \times \{0\})\,.$$

In class we saw two ways of generating a manifold.

(a) *Using a graph (Example 15).* Let $W \subset \mathbb{R}^k$ be open and $g \in C^l(W;\mathbb{R}^{n-k})$. Then the graph of g, defined as

$$M \;=\; G(g) \;:=\; \big\{(x, g(x)) : x \in W\big\}\,,$$

is a k-dim. C^l-MF.

(b) *As the preimage of a regular value (Example 16).* Let $\varphi \in C^l(U;\mathbb{R}^{n-k})$ for some open set $U \subset \mathbb{R}^n$. Then the set $M := \varphi^{-1}(\{0\})$ is a k-dim. C^l-MF if 0 is a regular value of φ (i.e. if $\operatorname{rank}\varphi'(p) = n-k$ for all $p \in M$).

1. In class we saw that the tangent space of a graph $M = G(g)$ as in (a) is given by $T_pM = \Phi'(x)\mathbb{R}^k$, where $\Phi(x) := (x, g(x))$ and $p = \Phi(x)$. Consider the special case $k = 1$ and $n = 2$, and verify that this expression for T_pM coincides with the expression for the tangent line, translated to pass through the origin, of the graph $y = g(x)$ that you learned in high school or in calculus.

2. Find the tangent space of the graph of the function $g(x, y) = x^2 + y^2\cos(x)$.

3. Let $M = \varphi^{-1}(\{0\})$ be the preimage of a regular value of $\varphi \in C^1$, as in (b) above. Prove the following fact mentioned but not proved in class: the tangent space T_pM is

$$T_pM \;=\; \operatorname{null}\varphi'(p)\,.$$

Hints. Using the implicit function theorem, express M locally as a graph $G(g)$. Differentiate the relation $\varphi(\Phi(x)) = 0$, where $\Phi(x) := (x, g(x))$, and use what you know about the tangent space of a graph (see the first sentence of Problem 1) to show that $T_pM \subset \operatorname{null}\varphi'(p)$. To conclude, find the dimensions of the spaces T_pM and $\operatorname{null}\varphi'(p)$.

4. (i) A *torus* is a doughnut-shaped surface in $\mathbb{R}^3$ that can be constructed as follows. Let $a > b > 0$ and consider the circle $\mathcal{C}$ of radius a in the xy-plane. By definition, the torus $T_{a,b}$ is the set of points (x, y, z) in $\mathbb{R}^3$ that lie at a distance b from the circle $\mathcal{C}$. Draw a sketch of $T_{a,b}$ and prove that it is a 2-dim. C^∞-MF.

Hint. To prove that $T_{a,b}$ is a manifold, it is easiest to exhibit it as the preimage of a regular value of a smooth function φ. To find such a φ, introduce the radius in the xy-plane, $r := \sqrt{x^2 + y^2}$, and argue geometrically in the zr-plane.

(ii) Find the tangent space of $T_{a,b}$ at a point $p = (x, y, z) \in T_{a,b}$.

5. The *special linear group* is defined as

$$\mathrm{SL}(n) \ := \ \left\{ X \in \mathbb{R}^{n\times n} : \det X = 1 \right\}.$$

(i) Prove that $\mathrm{SL}(n)$ is an (n^2-1)-dim. C^∞-MF in the space of $n \times n$ matrices.
Hint. According to (b) above, you have to prove that the derivative of $\varphi(X) := \det X$ has maximal rank, i.e. rank one. To differentiate the determinant, recall Problem 3.11(i).

(ii) Show that the tangent space $T_{I_n}\mathrm{SL}(n)$ is the space of matrices whose trace is zero.
Hint. Use Problem 3.

6. The *orthogonal group* is defined as

$$\mathrm{O}(n) \ := \ \left\{ X \in \mathbb{R}^{n\times n} : X^T X = I_n \right\}.$$

(i) Show that $\mathrm{O}(n)$ is an $\frac{n(n-1)}{2}$-dim. C^∞-MF in the space of $n \times n$ matrices.
Hints. Exhibit $\mathrm{O}(n)$ as the preimage of 0 of the function $\varphi(X) := X^T X - I_n$. Here $\varphi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$, where $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ is the space of symmetric $n \times n$ matrices. (Choosing this target space for φ is very important; otherwise you will not be able to satisfy the maximal rank condition.)
Compute the derivative $\varphi'(A)H$. To show that it has maximal rank, you have to show that the equation $\varphi'(A)H = S$ has a solution H for each $A \in \mathrm{O}(n)$ and $S \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.
You will also need to compute the dimension of $\mathbb{R}^{n\times n}_{\mathrm{sym}}$.

(ii) Show that the tangent space $T_{I_n}\mathrm{O}(n)$ is the space of antisymmetric matrices.

7.* Fix $h > 0$ and define the function $f : U \to \mathbb{R}^3$, where $U := (0, \infty) \times \mathbb{R}$ and

$$f(r, \theta) \ := \ (r\cos\theta, r\sin\theta, h\theta).$$

Sketch the set $M := f(U)$ and prove that it is a 2-dim. C^∞-MF.

Due: Thursday, May 9, in class.