Problem Set 4

1. Let $f \in C^{k+1}$ in a neighborhood of $a \in \mathbb{R}^n$. In class we saw that f can be expressed using its Taylor series as

$$f(a+h) \;=\; P_k(h) + R_k(h) \tag{1}$$

where

$$P_k(h) \;:=\; \sum_{r=0}^{k} \frac{1}{r!} D_h^r f(a)\,, \qquad R_k(h) \;=\; \frac{1}{(k+1)!} D_h^{k+1} f(\xi)\,,$$

for some ξ in the segment from a to $a+h$.

(i) Find the Taylor polynomial of degree 2 (i.e. $P_2(h)$) of the function $f(x,y) = \sin(xy)\mathrm{e}^{x^2}$ at $a = (0,0)$.

(ii) Find the Taylor polynomial of degree 3 (i.e. $P_3(h)$) of the function $f(x,y) = x^y$ at $a = (1,1)$.

(iii) Find the Taylor series (i.e. $\lim_{k\to\infty} P_k(h)$) of the function $f(x,y) = \sqrt{1+x^2+y^2}$ at $a = (0,0)$.

2. Locate the critical points and extrema of the functions

$$\text{(i)} \quad f(x,y) \;=\; x^3 + y^3 + 3xy\,, \qquad \text{(ii)} \quad f(x,y,z) \;=\; x^2+y^2+z^2-2xyz\,.$$

3. (i) In class we proved that $R_k(h) = O(|h|^{k+1})$, so that for $f \in C^{k+1}$ we have

$$f(a+h) \;=\; P_k(h) + O(|h|^{k+1})\,. \tag{2}$$

In some applications we want to make the weaker assumption $f \in C^k$. Prove, using Taylor's theorem, that if $f \in C^k$ then

$$f(a+h) \;=\; P_k(h) + o(|h|^k)\,. \tag{3}$$

Note that (3) is weaker than (2), but for almost all practical purposes just as useful; its advantage is that it holds under the weaker assumption $f \in C^k$ rather than $f \in C^{k+1}$. *Hints.* Start from (1) with k replaced by $k-1$. You have to prove that

$$R_{k-1}(h) \;=\; \frac{1}{k!} D_h^k f(a) + o(|h|^k)\,.$$

(ii) Use (i) to prove the following, stronger, version of the criterion for locating local extrema proved in class.

Suppose that f is C^2 in a neighborhood of $a \in \mathbb{R}^n$, and that a is a critical point of f. Then

$$\begin{aligned} f''(a) \text{ positive definite} &\quad\Longrightarrow\quad a \text{ is a local minimum of } f \\ f''(a) \text{ negative definite} &\quad\Longrightarrow\quad a \text{ is a local maximum of } f \\ f''(a) \text{ nondefinite} &\quad\Longrightarrow\quad a \text{ is neither a local maximum nor a local minimum}\,. \end{aligned}$$

(Note that this result is stronger because we only require that f be C^2 and not C^3.)

Hints. The proof is analogous to the one given in class, except that you should use (3) instead of (2).

4. Let $A, B \subset \mathbb{R}^n$ be open, and suppose that $f : A \to B$ such that both f and f^{-1} are C^1. Prove that $f'(a)$ is an invertible matrix for all $a \in A$, and that for all $a \in A$ with $b := f(a)$ we have

$$(f^{-1})'(b) \;=\; (f'(a))^{-1}\,.$$

(In words: the derivative of the inverse is the inverse of the derivative.)

5. Prove that the function $f(x) := |x|$ is differentiable on $\mathbb{R}^n \setminus \{0\}$ and find $\nabla f(x)$.

6. The Laplacian Δ is defined by

$$\Delta f(a) \;:=\; \operatorname{Tr} f''(a) \;=\; \sum_{i=1}^n D_i^2 f(a)\,.$$

It appears in just about every equation describing a law of nature, such as heat conduction, motion of waves, motion of quantum-mechanical particles, electric and magnetic fields, and Brownian motion.

(i) Prove that Δ is *invariant under rotations* in the following sense. Let $v_1, \dots, v_n$ be an orthonormal basis of $\mathbb{R}^n$. Then

$$\Delta f \;=\; \sum_{i=1}^n D_{v_i}^2 f$$

for all $f \in C^2(\mathbb{R}^n)$.

(ii) Prove that for $f, g \in C^2$ we have

$$\Delta(fg) \;=\; f\Delta g + 2\langle \nabla f\,, \nabla g\rangle + g \Delta f\,.$$

(iii) Let $n = 2$ and consider the polar coordinates defined in class:

$$T : [0,\infty) \times [0, 2\pi) \to \mathbb{R}^2\,, \qquad T(r,\theta) \;:=\; \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}\,.$$

Express Δ in polar coordinates. More precisely, let $f \in C^2(\mathbb{R}^2)$ and define $g := f \circ T$. Show that for $r > 0$ we have

$$(\Delta f)(T(r,\theta)) \;=\; \biggl(D_r^2 g + \frac{1}{r^2} D_\theta^2 g + \frac{1}{r} D_r g\biggr)(r,\theta)\,.$$

Hints. You will have to use the chain rule. One way to start is to compute $D_r g$ and $D_\theta g$; you can differentiate these expressions again to obtain $D_r^2 g$ and $D_\theta^2 g$.

(iv) Suppose that $f : \mathbb{R}^n \to R$ is invariant under rotations in the sense that $f(x) = g(|x|)$ for some $g \in C^2((0,\infty))$. Prove that

$$\Delta f(x) \;=\; g''(|x|) + \frac{n-1}{|x|} g'(|x|)\,.$$

(v) Let

$$\psi : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}\,, \qquad \psi(t,x) \;:=\; \frac{1}{t^{n/2}} \exp\biggl(-\frac{|x|^2}{4t}\biggr)\,.$$

Using (iv) show that

$$D_t \psi(t,x) \;=\; \Delta\psi(t,x) \qquad (t > 0\,,\; x \in \mathbb{R}^n)\,, \tag{4}$$

where Δ is the Laplacian in the variables $x_1, \dots, x_n$ (and not t).

The equation (4) is call the *heat equation* and its solutions, such as ψ, model the diffusion of heat through a conducting medium. The solution ψ given above corresponds to a single heat source at $x = 0$ when $t = 0$, which diffuses through space as t increases. (If you want, you can try to plot ψ for $n = 1$ for various values of $t > 0$ to get an idea of how the heat distribution evolves in time.)

(vi) Let $c > 0$ and fix a unit vector $v \in \mathbb{R}^n$ and $f \in C^2(\mathbb{R})$. Show that

$$\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\,, \qquad \psi(t,x) \;:=\; f\bigl(\langle x, v\rangle - ct\bigr)$$

satisfies the *wave equation*

$$D_t^2 \psi(t,x) \;=\; c^2 \Delta\psi(t,x)\,. \tag{5}$$

(As above, the Laplacian Δ only acts on $x_1, \dots, x_n$ and not t.) As its name implies, solutions ψ of the wave equation describe the motion of waves (e.g. sound, light) through space; the speed of the waves (speed of sound or speed of light) is c.

7. It is of fundamental importance for many arguments in analysis that there exists a C^∞ function which is positive inside the unit ball and zero outside. An example is

$$f(x) \;:=\; \begin{cases} \exp\bigl(1/(|x|^2 - 1)\bigr) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1\,. \end{cases}$$

Prove that $f \in C^\infty(\mathbb{R}^n)$.

Hints. Write $f(x) = g(|x|^2)$ with some function (that you should determine) $g : \mathbb{R} \to \mathbb{R}$. Show that it suffices to show that $g \in C^\infty(\mathbb{R})$. In order to prove that $g \in C^\infty(\mathbb{R})$, the following fact should be helpful: for all $k \in \mathbb{N}$ and $r < 1$, we have

$$g^{(k)}(r) = \frac{P_k(r)}{Q_k(r)} \exp\left(\frac{1}{r-1}\right),$$

where P_k and Q_k are polynomials. Prove this fact by induction on k.

8. A linear transformation $A : R^n \to \mathbb{R}^m$ is called *conformal* if there is a number $\rho > 0$ such that $A^T A = \rho \mathbb{1}$.

(i) Prove that a conformal transformation preserves angles in the sense that, for any nonzero $v, w \in \mathbb{R}^n$, the vectors Av and Aw are also nonzero and the angle between v and w is the same as the angle between Av and Aw.

(ii)* Prove that a linear map that preserves angles in the sense given in (i) is conformal.

(From (i) and (ii) we deduce that a linear map is conformal if and only if it preserves angles.)

(iii) A mapping $f \in C^1(A; \mathbb{R}^m)$, where $A \subset \mathbb{R}^n$, is called *conformal* at $a \in A$ if $f'(a)$ is conformal.

Let γ_1 and γ_2 be two C^1 curves in $\mathbb{R}^n$ that intersect: $\gamma_1(0) = \gamma_2(0)$. Suppose that f is conformal at this point of intersection. Prove that the angle between the tangents of γ_1 and γ_2 at time 0 is the same as the angle between the tangents of $f \circ \gamma_1$ and $f \circ \gamma_2$ at time 0.

(iv)* The *stereographic projection* is a projection used to map the unit sphere in $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. It is defined as follows. Let

$$\mathbb{S}^n := \{u \in \mathbb{R}^{n+1} : |u| = 1\}$$

be the unit sphere, and let $p := (0, \ldots, 0, 1)^T$ be the "north pole" of the sphere. For $x \in \mathbb{R}^n$, let $l(x)$ be the line in $\mathbb{R}^{n+1}$ that passes through the two points $(x, 0)$ and p.

Draw a sketch of $\mathbb{R}^{n+1}$, along with $\mathbb{S}^n$ and $l(x)$. (You will have to take $n = 1$ or $n = 2$ for the sketch).

Show that $l(x)$ intersects $\mathbb{S}^n$ at a unique point, which we denote by $S(x)$. Show also that S is a bijection between $\mathbb{R}^n$ and $\mathbb{S}^n \setminus \{p\}$. It is called the *stereographic projection.*

Prove that

$$S(x) = \frac{1}{1 + |x|^2} \begin{pmatrix} 2x \\ |x|^2 - 1 \end{pmatrix}.$$

Finally, prove that S is conformal.

This fact is of great interest to cartographers: using S we may represent the surface of the earth on a flat piece of paper in such a way that all angles are preserved. The downside is that areas are not preserved; in fact, there is no projection that preserves both angles and areas (this is a mathematical theorem). The stereographic projection is accurate near the "south pole" $(0, \ldots, 0, -1)^T$, and becomes increasingly distorted as one approaches the north pole p.

Due: Thursday, April 11, in class.