Problem Set 3

1. Show that the following functions are differentiable and compute their differentials.

$$\text{(i)}\quad f(x,y,z) \;=\; \begin{pmatrix} xy^3 \\ z\sin y \\ x^2 - y^2 z \end{pmatrix}, \qquad \text{(ii)}\quad f(x,y) \;=\; \begin{cases} \frac{x^3}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

2. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and homogeneous of degree $\alpha \in \mathbb{R}$, i.e. $f(tx) = t^\alpha f(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$. Prove that $f'(x)x = \alpha f(x)$. (This is sometimes known as Euler's identity.)

3. Prove that a continuously differentiable function (i.e. a function whose partial derivatives exist and are continuous) on $\mathbb{R}^n$ is Lipschitz continuous on any compact subset of $\mathbb{R}^n$.

4.
(i) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and γ a differentiable path such that $f(\gamma(t))$ is constant. Prove that $\nabla f(\gamma(t))$ is orthogonal to $\gamma'(t)$ for all t.

(ii) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$. The rate of growth of f in the direction $v \in \mathbb{R}^n$ is given by the directional derivative $D_v f(a)$. Show that direction of maximal growth, i.e. the unit vector v for which $D_v f(a)$ is maximal, is $\nabla f(a)/|\nabla f(a)|$.

(iii) Interpret (i) and (ii) in terms of the following scenario: you are hiking in mountainous terrain, and $f(x,y)$ represents the height of the terrain. If you are in possession of a map that includes contours lines, how should you walk if you want to reach a nearby summit as quickly as possible? Illustrate your argument using a sketch.

5. Recall that in class a *path* was defined to be continuous, piecewise continuously differentiable map $\gamma : [a,b] \to \mathbb{R}^n$. In other words, there is a partition $a = a_0 < a_1 < \cdots < a_k = b$ such that γ is continuously differentiable on (a_i, a_{i+1}) for each $i = 0, \dots, k-1$. Recall also that a 1-form λ is a continuous map from $\mathbb{R}^n$ to the space of $1 \times n$ matrices (dual vectors). The path integral of λ along γ was defined as

$$\int_\gamma \lambda \;:=\; \int_a^b \lambda(\gamma(t))\gamma'(t)\,\mathrm{d}t\,.$$

Finally, recall that the reversal of $\gamma : [0,1] \to \mathbb{R}^n$ was defined through $(-\gamma)(t) := \gamma(1-t)$, and the join of the two paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$ satisfying $\gamma_1(1) = \gamma_2(0)$ was defined as the path $(\gamma_1 \oplus \gamma_2) : [0,2] \to \mathbb{R}^n$ given by

$$(\gamma_1 \oplus \gamma_2)(t) \;:=\; \begin{cases} \gamma_1(t) & \text{if } t \in [0,1] \\ \gamma_2(t-1) & \text{if } t \in (1,2]\,. \end{cases}$$

Prove that

$$\int_{-\gamma} \lambda \;=\; -\int_{\gamma} \lambda\,, \qquad \int_{\gamma_1 \oplus \gamma_2} \lambda \;=\; \int_{\gamma_1} \lambda_1 + \int_{\gamma_2} \lambda\,.$$

Hint. If you are having trouble with general piecewise differentiable paths, try first proving this for differentiable paths. Then extend your result to arbitrary piecewise differentiable paths.

6. If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $f(0) = 0$, prove that there exist continuous $g_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) \;=\; \sum_{i=1}^{n} x_i g_i(x)\,.$$

Hint. Write $f(x) - f(0) = f(tx)|_{t=0}^{1}$ and apply the fundamental theorem of calculus.

7. In this problem we work in $\mathbb{R}^2$. Define the 1-forms

$$\mu(x,y) \;:=\; (0,x)\,, \qquad \nu(x,y) \;:=\; (-y,0)\,, \qquad \lambda(x,y) \;:=\; \frac{1}{2}(-y,x)\,.$$

(i) Let γ be the path that traces the boundary of a disk of radius r once in the counterclockwise direction. Compute $\int_\gamma \mu$, $\int_\gamma \nu$, and $\int_\gamma \lambda$. Do the same when γ traces the boundary of a rectangle with side lengths a and b. What do you observe?

(ii) Let γ be an arbitrary closed path. Prove that $\int_\gamma \mu = \int_\gamma \nu = \int_\gamma \lambda$.
Hint. For e.g. the first equality, find a function f such that $\mu = \nu + Df$.

(iii)* In general, if γ is a closed path that traces the boundary of an arbitrary open region A in the counterclockwise direction, then any of the above integrals gives the area of A.

Prove this fact under the assumption that A is *star shaped*, i.e. that for any $x \in A$, the segment joining x to the origin is contained in A.

Hints. Consider the triangle with vertices $(0,0)$, (x,y), and $(x+\Delta x, y+\Delta y)$, and suppose that $x\,\Delta y - y\,\Delta x \geqslant 0$. What does this condition mean geometrically? (Using the vector product in $\mathbb{R}^3$ might be helpful here.) Prove that the area of this triangle is

$$\frac{x\,\Delta y - y\,\Delta x}{2}\,.$$

Prove that by assumption on γ, the three vertices 0, $\gamma(t)$, and $\gamma(t) + \gamma'(t)\,\Delta t$ satisfy the above condition for any t and $\Delta t > 0$. Then express the area of A using a Riemann sum, by breaking it up into thin triangular slices.

8. In $\mathbb{R}^3$ it is often convenient to use *spherical coordinates* $(r,\theta,\phi) \in [0,\infty) \times [0,\pi] \times [0,2\pi)$. The coordinate map is $(x,y,z)^T = T(r,\theta,\phi)$, where

$$T(r,\theta,\phi) \;:=\; \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}\,.$$

(i) Give a geometric interpretation of the parameters r, θ, and ϕ.

(ii) Compute T'. Show that the columns of T' are orthogonal. Interpret this result geometrically using a sketch.

(iii) Let f be differentiable on $\mathbb{R}^n$, and define $g := f \circ T$. The function g represents the function f expressed in spherical coordinates. Compute all partial derivatives of g in terms of the partial derivatives of f. Find $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial \theta}$ for the functions $f(x, y, z) = x^2 + y^2 + z^2$ and $f(x, y, z) = x - y$.

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in n variables, i.e.

$$f(x_1, \ldots, x_n) \;=\; \sum_{k_1,\ldots,k_n=0}^{m} a_{k_1\ldots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

for some coefficients $a_{k_1\ldots k_n}$. Prove that f is differentiable.

10. Let

$$U \;:=\; \left\{A \in \mathbb{R}^{n\times n} : A \text{ is an invertible matrix}\right\}.$$

(i) Show that U is an open subset of $\mathbb{R}^{n\times n}$.
Hint. Use that $A \in U$ if and only if $\det(A) \neq 0$. Prove that det is a continuous function on $\mathbb{R}^{n\times n}$. Recall that a function is continuous if and only if the preimages of open sets are open.

(ii) Prove that the map $f : U \to U$ defined by $f(A) := A^{-1}$ is differentiable with

$$Df_A(B) \;=\; -A^{-1}BA^{-1}\,. \tag{1}$$

Hints. In order to prove that f is differentiable, you can use e.g. Cramer's rule to show that all entries of f have continuous partial derivatives in A. In order to obtain (1), it is easiest to compute the directional derivative $\frac{\mathrm{d}}{\mathrm{d}t} f(A + tB)|_{t=0}$ from the identity $Af(A) = \mathbb{1}$ for all $A \in U$.

11. (i) Prove that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is differentiable. If A is invertible, prove that the differential of det at A is given by

$$D\det{}_A(B) \;=\; \mathrm{Tr}(A^{-1}B)\det(A)\,.$$

Hints. Work directly using the definition

$$\det(A) \;=\; \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}\,.$$

For the second part, assume first that $A = \mathbb{1}$ and compute the directional derivative $\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathbb{1} + tB)|_{t=0}$. In a second step, take a general invertible A and reduce the problem to the first case.

(ii)* For any $n \times n$ matrix A we define the exponential

$$\exp(A) \;:=\; \sum_{k=0}^{\infty} \frac{1}{k!} A^k \,.$$

Prove that this series converges absolutely (componentwise).
Hint. Introduce $M := \max_{i,j} |A_{ij}|$ and estimate $|(A^k)_{ij}| \leqslant M^k n^{k-1}$.

(iii)* Show that $\exp(tA)\exp(sA) = \exp((t+s)A)$.
Hint. Using the fact that both series converge absolutely, you may multiply the series out and rearrange the terms.

(iv)* Prove that $\exp(At)$ is differentiable in t with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(At) \;=\; A \exp(At) \,.$$

(v)* By (iii), $\exp(tA)$ is invertible for all t (why?). Use (i) to show

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(\exp(tA)) \;=\; \mathrm{Tr}(A) \det(\exp(tA)) \,.$$

Solve the differential equation to conclude that

$$\det(\exp(A)) \;=\; \exp(\mathrm{Tr}\, A) \,.$$

This problem is an example of how analysis can be used to derive identities in linear algebra.

(vi)** It is not hard to see that $\exp : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is differentiable. Can you find its differential?

Due: Tuesday, March 26, in class.