Problem Set 2

1. Prove that a uniformly continuous function is continuous.

2. Does the function

$$f(x,y) = \frac{x^3 - y^3}{|x-y| + y^2}$$

have a limit as $(x,y) \to 0$? If yes, give the limit. Answer the same question for the function

$$f(x,y) = \left(x + y^3 \, , \, x + \frac{x}{x^2+y^2} \right).$$

3. Recall the topological characterization of continuity: f is continuous if and only if $f^{-1}(U)$ is open whenever U is open. Here f^{-1} cannot be replaced with f: find a continuous function f and an open set U such that $f(U)$ is not open.

4. A function $f : D \to \mathbb{R}^m$ for $D \subset \mathbb{R}^n$ is called *Lipschitz continuous* if there exists a constant L such that

$$|f(x) - f(y)| \leqslant L|x-y| \qquad \forall x, y \in D\,.$$

The constant L is called the *Lipschitz constant of* f.

(i) Prove that a Lipschitz continuous function is uniformly continuous.

(ii) Find an example of a uniformly continuous function that is not Lipschitz continuous.

(iii) Prove that the distance (or norm) $x \mapsto |x|$ is a Lipschitz continuous function from $\mathbb{R}^n$ to $\mathbb{R}$.

(iv) Prove that addition $(x,y) \mapsto x+y$ is a Lipschitz continuous function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$.

(v) Prove that the scalar product $(x,y) \mapsto \langle x\,, y \rangle$ is a Lipschitz continuous function from $D \times D$ to $\mathbb{R}$ for any bounded domain D.

5. The notion of *Hölder continuity* is a generalization of Lipschitz continuity. Let $\alpha \geqslant 0$. We say that $f : D \to \mathbb{R}^m$ is *α-Hölder continuous* if there exists a constant L such that

$$|f(x) - f(y)| \leqslant L|x-y|^\alpha \qquad \forall x, y \in D\,.$$

(i) What does this condition mean if $\alpha = 1$? What about $\alpha = 0$?

(ii) Prove that an α-Hölder continuous function is uniformly continuous provided $\alpha > 0$.

(iii) Prove that if D is bounded, $\alpha \leqslant \beta$, and f is β-Hölder continuous, then f is α-Hölder continuous.

(iv) Prove that if f is α-Hölder continuous for $\alpha > 1$, then f is constant.

Hints. Pick $x, y \in D$ and estimate $|f(x) - f(y)|$ by choosing a path $(x_0, \ldots, x_n)$ defined by

$$x_i := x + \frac{i}{n}(y - x)\,.$$

Write $f(y) - f(x)$ as a telescoping sum. In the end take the limit $n \to \infty$.

6. A subset $A \subset \mathbb{R}^n$ is *dense* if for every $x \in \mathbb{R}^n$ the set $A \cap B_\varepsilon(x)$ is not empty.

(i) Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(ii) Using (i), prove that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

(iii) Let f and g be continuous functions and A be a dense set in $\mathbb{R}^n$. Prove that if f and g coincide on A then $f = g$ (i.e. they coincide on $\mathbb{R}^n$).

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined through

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

Show that f does not have a limit at any point in $\mathbb{R}$.

8. Recall that the Bolzano-Weierstrass theorem states that any bounded sequence in $\mathbb{R}$ has a convergent subsequence. Use this result to prove that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. (This fact was used in class in the proof of the sequential characterization of compactness.)

Hint. Let $(x_k)_{k \in \mathbb{N}}$ be such a sequence, and apply the one-dimensional result to its components x_k^i, where $i = 1, \ldots, n$. You will have to extract n decreasing subsequences.

9. Recall that the Heine-Borel characterization of compactness says that A is compact if and only if for any family of open sets $(A_s)_{s \in S}$ that cover A (an "open cover") there exists a finite subset $S_0 \subset S$ such that the finite family of open sets $(A_s)_{s \in S_0}$ cover A (a "finite subcover").

(i) Find an example of a bounded set together with an open cover which has no finite subcover.

(ii) Find an example of a closed set together with an open cover which has no finite subcover.

10. Let f be defined by

$$f(x, y) = \begin{cases} |y/x^2|\, \mathrm{e}^{-|y/x^2|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$$

Prove that f is discontinuous at $(0, 0)$. Prove that f is continuous along any line passing through the origin.

11.* (This problem is optional and will not influence your homework score. It's a good one, though.) On $\mathbb{R}$ we define the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = p/q \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ having no common divisor}. \end{cases}$$

Prove that f is continuous at every irrational point and discontinuous at every rational point.

Due: Thursday, February 28, in class.