Problem Set 1

1. (i) Prove that for any sets A, B, and C we have

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

These identities may be interpreted as the mutual distributivities of $\cap$ and $\cup$.

Hint. You can use e.g. a Venn diagram, but you should understand why this constitutes a rigorous proof.

(ii) Let S be an index set and A_s a family of subsets of some set X. Prove *De Morgan's laws*

$$\left(\bigcup_{s \in S} A_s\right)^c = \bigcap_{s \in S} A_s^c, \qquad \left(\bigcap_{s \in S} A_s\right)^c = \bigcup_{s \in S} A_s^c,$$

where $A^c := X \setminus A$ denotes the complement of A.

2. Show that the scalar product $\langle \cdot , \cdot \rangle$ on $\mathbb{R}^n$ satisfies the *parallelogram law* or *polarization identity*

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2.$$

What does this identity mean geometrically for the parallelogram spanned by x and y (i.e. the parallelogram with vertices 0, x, y, and $x + y$)?

3. (i) Prove that if A and B are closed then $A \cup B$ is also closed. Find an example that shows that infinite unions of closed sets are not necessarily closed.

(ii) Prove that if for each $s \in S$ the set A_s is closed then $\bigcap_{s \in S} A_s$ is also closed.

4. For any function $f : X \to Y$ and $A \subset Y$ recall the preimage map $f^{-1}(A) := \{x \in X : f(x) \in A\}$.

(i) Show that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \qquad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

(ii) Show that

$$f(A \cup B) = f(A) \cup f(B), \qquad f(A \cap B) \subset f(A) \cap f(B).$$

Find an example to show that in general $f(A \cap B)$ is not equal to $f(A) \cap f(B)$. Show that if f is injective (or one-to-one) then we have

$$f(A \cap B) = f(A) \cap f(B).$$

Hint. It may be helpful to write for instance

$$f(A \cap B) = \big\{y \in Y : \exists x \in A \cap B : y = f(x)\big\}.$$

5. (i) Let $A_1, A_2, A_3, \ldots$ be subsets of some set X, and define

$$U := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\,, \qquad V := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k\,.$$

Which one of $U \subset V$ and $V \subset U$ is true? Prove your claim.

(ii) Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions on some set X, and f a function on X. Show that the *set of convergence*

$$C := \Big\{x \in X : \lim_{k\to\infty} f_k(x) = f(x)\Big\}$$

may be written as

$$C = \bigcap_{l=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \Big\{x \in X : |f_k(x) - f(x)| \leqslant \frac{1}{l}\Big\}\,.$$

6. Prove that a sequence in $\mathbb{R}^n$ converges if and only if all of its components converge.

7. The *closure* of a set A is by definition

$$\overline{A} := \bigcap_{\substack{B \supset A:\\ B \text{ closed}}} B\,.$$

(i) Prove that $\overline{A}$ is closed. Hence, $\overline{A}$ is the smallest closed set containing A.

(ii) Recall that by definition x is a *limit point* of A if for all $r > 0$ the set $B_r(x)$ contains a point of A that is not x. Show that

$$\overline{A} = A \cup \{\text{limit points of } A\}\,.$$

Hints. This is the same as proving $\overline{A} = L$ where

$$L := \Big\{\lim_{k\to\infty} x_k : (x_k)_{k\in\mathbb{N}} \text{ is a convergent sequence with } x_k \in A\ \forall k \in \mathbb{N}\Big\}$$

(why?). In order to prove the equality, you have to prove that $\overline{A} \subset L$ and $L \subset \overline{A}$. For the first inclusion, take $y \in \overline{A}$ and prove, by contradiction (i.e. using the method of indirect proof) that for all $r > 0$ we have $B_r(y) \cap A \neq \emptyset$. Once this is proved, it will easily follow that $y \in L$.

(iii) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Hint. Use the characterization of (ii).

(iv) Find an example that shows that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ in general.

8. (i) Prove the following characterization of continuity, which is a (stronger) variant of the one given in class: a function f is continuous at a if and only if every sequence $(x_k)_{k\in\mathbb{N}}$ that converges to a has a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} f(x_{k_j}) = f(a)$.

Hint. The "only if" direction follows from the characterization of continuity given in class. In order to prove the "if" direction, do an indirect proof by assuming that f is not continuous at a.

(ii) Suppose that f is a continuous, bijective function defined on a compact set A. Show that f^{-1} is also continuous. (A bijective continuous function whose inverse is also continuous is called a *Homeomorphism.*)

Hint. By (i), it suffices to show (why?) that if $(x_k)_{k\in\mathbb{N}}$ is a sequence in A satisfying $f(x_k) \to f(x)$ for some $x \in A$, then there exists a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $x_{k_j} \to x$.

(iii) Can you find an example of a continuous, bijective function f such that f^{-1} is not continuous?

Due: Thursday, February 14, in class.