Practice Final

The questions are representative of the kind of questions to expect on the final. The length, however, is not: the actual final will be shorter in accordance with the 110 minutes that you will have. While doing the practice final, watch out that you don't take too much time on any problem, and make sure that your writing is clear.

1. Let X be the set of sequences $a = (a_k)_{k \in \mathbb{N}}$, where $a_k \in \mathbb{R}^n$, such that $\sum_{k=1}^{\infty} |a_k|^2$ converges. Prove that X is a vector space.

Hint. You will need the inequality $xy \leqslant x^2/2 + y^2/2$ for $x, y \in \mathbb{R}$.

2. Do each problem that is possible; if the problem is not possible, explain why.

(i) Give a family of open sets whose intersection is closed.

(ii) Give a family of closed sets whose intersection is open.

(iii) Give a family of compact sets whose union is bounded and not compact.

(iv) Give a family of compact sets whose intersection is not compact.

3. Consider the following four properties of a function: (1) continuous, (2) uniformly continuous, (3) Lipschitz continuous, (4) continuously differentiable.

(i) For $\ell = 1, 2, 3$ give an example of a function that has the property (ℓ) but not $(\ell + 1)$.

(ii) Now consider functions defined on a compact set. Give all implications among the four propreties (1)–(4). (In other words, give all instances of $\ell, \ell' \in \{1, 2, 3, 4\}$ such that property (ℓ) implies property (ℓ').)

4. Let A be an antisymmetric $n \times n$ matrix. Suppose that $\gamma \in C^1(\mathbb{R}; \mathbb{R}^n)$ satisfies

$$\gamma'(t) \;=\; A\gamma(t)\,.$$

Prove that $|\gamma(t)|$ is constant in t.

5. Let $f \in \mathbb{C}^1(\mathbb{R}^{n \times n}; \mathbb{R}^{n \times n})$ be defined through

$$f(X) \;:=\; X^k\,,$$

where $X \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix and $k \in \mathbb{N}$ is fixed. Find the differential $\mathrm{d}f$.

6. Is the vector field

$$V(x,y) \;:=\; \begin{pmatrix} y\cos x + y^2 \\ xy + \sin x \end{pmatrix}$$

conservative?

7. Let $f \in C^1(\mathbb{R}^n;\mathbb{R})$ and suppose that $|f'(x)| \leqslant C$ for all $x \in \mathbb{R}^n$. Prove that

$$|f(x) - f(y)| \;\leqslant\; C|x-y|$$

for all $x, y \in \mathbb{R}^n$.

8. Does the function

$$f(x,y) \;:=\; \frac{(x^2-y^2)^2(x+y)}{(x-y)^2 + x^2 + y^2}$$

have a limit as $(x,y) \to (0,0)$?

9. Let $f \in C(\mathbb{R}^n)$. Define the new function

$$g(x) \;:=\; \frac{1}{\mu(B_1(x))} \int_{B_1(x)} f \, \mathrm{d}\mu\,.$$

Thus, $g(x)$ is the average value of f over the ball $B_1(x)$ of radius 1 centered at x. Prove that g is continuous.

Hint. One way to do this problem is to estimate $|g(x) - g(y)|$ by rewriting $g(y)$ as an integral over $B_1(x)$ after a change of variables.

10. Give the Taylor polynomial P_4 of the function $f(x,y) = \mathrm{e}^{xy}$ around $a = (0,0)$.

11. Find and classify all local extrema of the function

$$f(x,y) \;=\; \sin x \, \sin y\,.$$

12. Let $p \in (1,\infty)$ and define the set

$$M \;:=\; \big\{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p = 1\big\}\,.$$

(i) Sketch this set for $n=2$ and a couple of values of p. What happens as $p \to \infty$?

(ii) Find all critical points of the function $|x|^2$ on M. You do not have to determine the nature of these critical points.

(iii) Prove that M is an $(n-1)$-dimensional C^1-manifold.

(iv) Find T_xM for $x \in M$.

13. Find the tangent space at $(x,y) = (0,0)$ of the graph of the function

$$g(x,y) \;:=\; \mathrm{e}^{x-y}\sin(x+y)\,.$$

14. Consider the system of constraints

$$\begin{aligned}\varphi_1(x, y_1, y_2) &= x^3 + y_1^3 + y_2^3 - 7 = 0\,,\\ \varphi_2(x, y_1, y_2) &= xy_1 + y_1y_2 + y_2x + 2 = 0\,.\end{aligned}$$

Show that the set of points (x, y_1, y_2) satisfying both of these constraints can be locally expressed as a graph of an explicit function $(y_1, y_2) = f(x)$ around the point $(2, -1, 0)$. Compute $f'(2)$.

15. Prove the inverse function theorem starting from the implicit function theorem.

16. Suppose that $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is *linear*. Prove the implicit function theorem for φ from first principles.

17. Prove that there is a C^1 square root function for $n \times n$ matrices in a neighborhood of the identity. More precisely, prove that there exists a neighborhood $U \ni I_n$ and a function $f \in C^1(U; \mathbb{R}^{n\times n})$ such that $f(X)^2 = X$ for all $X \in U$.

Hint. Apply the inverse function theorem to the function $g(X) := X^2$ around the point $X = I_n$.

18. Recall the *spherical coordinates* $(r, \theta, \phi) \in (0, \infty) \times (0, \pi] \times (0, 2\pi)$, whose coordinate map is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T(r, \theta, \phi) := \begin{pmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix}.$$

Calculate $\det T'$, and use this information to calculate the volume of a sphere of radius R using spherical coordinates.

19*. Give a bounded set A such that A is not (Jordan-)measurable but $\overline{A}$ is.

20. Find the area of

$$A := \{(x, y) \in \mathbb{R}^2 : x > 0\,,\, a < y/x < b\,,\, c < xy < d\}\,.$$

Hint. Use the new variables $u := y/x$ and $v := xy$.