Please provide complete and well-written solutions to the following exercises.

Due June 4, in the discussion section.

Assignment 8

Exercise 1. Section 6.1, Exercise 8 in the textbook.

Exercise 2. Let $x, y \in \mathbf{R}$. Verify that $|x+y| \leq |x|+|y|$. Deduce that the triangle inequality holds for the 1-norm and the ∞-norm on $\mathbf{R}^n$

Exercise 3. Using the axioms for inner product spaces, show the following things:

(e) For all $v, v', w \in V$, $\langle w, v+v' \rangle = \langle w, v \rangle + \langle w, v' \rangle$. (Linearity in the second argument)
(f) For all $v, w \in V$, for all $\alpha \in \mathbf{F}$, $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$.
(g) For all $v \in V$, $\langle v, 0 \rangle = \langle 0, v \rangle = 0$.
(h) $\langle v, v \rangle = 0$ if and only if $v = 0$.

Exercise 4. Let $i := \sqrt{-1}$. Consider $W \subseteq \mathbf{C}^3$ defined by $W := \{(1,0,i),(1,2,1)\}$. Find $W^{\perp}$.

Exercise 5. Let V be an inner product space.

- Show that $\{0\}^{\perp} = V$ and $V^{\perp} = \{0\}$.
- Let V_1 be a subspace of an inner product space V. Show that $V_1^{\perp}$ is a subspace of V.
- Let S, T be subsets of V. If $S \subseteq T$, show that $T^{\perp} \subseteq S^{\perp}$.

Exercise 6. Let $V = C([0,1])$ be the space of continuous real valued functions on the interval $[0,1]$. Let $f, g \in C([0,1])$ and define the inner product $\langle f, g \rangle := \int_0^1 f(t)g(t)dt$. Let W be the subspace spanned by the linearly independent set $(t, \sqrt{t})$. Find an orthonormal basis for W. Consider $h(t) = t^2$. Find $w \in W$ such that $\langle h-w, h-w \rangle$ is as small as possible.

Exercise 7. Section 6.3, Exercise 2 in the textbook.

Exercise 8. Let $V, \|\cdot\|_V$ and let $W, \|\cdot\|_W$ be two n-dimensional normed linear spaces over $\mathbf{R}$. We know that V, W are then isomorphic as vector spaces. We now introduce a much stronger concept. We say that V, W are **isometric** if there exists a linear map $T \colon V \to W$ such that T is one-to-one and onto, and such that for all $v \in V$, $\|T(v)\|_W = \|v\|_V$.

(a) Assume additionally that V, W are n-dimensional inner product spaces. Show that V, W are then isometric.
(b) Find two 2-dimensional normed linear spaces V, W over $\mathbf{R}$ that are not isometric.

So, the notion of isometry has no real use in telling two different inner product spaces apart. But this notion is very useful in telling two different normed linear spaces apart.

Exercise 9. Let A be an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $\det(A) = \prod_{i=1}^{n} \lambda_i$ and $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$.

Exercise 10. Find a 2×2 matrix A such that A has eigenvectors $(1, 1)$ and $(1, 0)$, such that $A \neq I_2$, and such that $A^2 = I_2$. Recall that I_2 is the 2×2 identity matrix. (Hint: It may be helpful to use Section 5.1, Exercise 15 in the textbook, which we did last week for homework. This should help you identify what eigenvalues A could have.)

Exercise 11. Section 6.4, Exercise 2(a) in the textbook.

Exercise 12. Let $\mathbf{F}$ denote $\mathbf{R}$ or $\mathbf{C}$. Let $T: V \to W$, $S: V \to W$ and let $R: U \to V$ be linear transformations between inner product spaces U, V, W over $\mathbf{F}$. Verify the following facts

(a) $(T + S)^* = T^* + S^*$.
(b) For all $\alpha \in \mathbf{F}$, $(\alpha T)^* = \overline{\alpha} T^*$.
(c) $(T^*)^* = T$.
(d) $(TR)^* = R^* T^*$.
(e) If T is invertible, then $(T^{-1})^* = (T^*)^{-1}$.

Exercise 13. Let A be an $m \times n$ matrix. Show that $\mathrm{rank}(A) = \mathrm{rank}(A^\dagger)$.

Exercise 14. Let A be an $n \times n$ matrix with elements in $\mathbf{C}$. Show that $\det(A^\dagger) = \overline{\det(A)}$.