Please provide complete and well-written solutions to the following exercises.

Due May 14, in the discussion section.

Assignment 6

Exercise 1. section 3.2, Exercise 1(cdefhi) in the textbook.

Exercise 2. Compute the rank of the following matrix, then compute the inverse of the following matrix if it exists.

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Exercise 3. section 3.2, Exercise 6(ade) in the textbook.

Exercise 4. section 4.2, Exercise 1(abcdefgh) in the textbook.

Exercise 5. section 4.4, Exercise 1(cdefgik) in the textbook.

Exercise 6. Find matrices A, B such that $\det(A + B) \neq \det(A) + \det(B)$

Exercise 7. Let $n \in \mathbf{N}$, and let $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a one-to-one and onto map. Then σ is called a **permutation** on n elements. The set of all permutations on n elements is denoted by S_n. The set S_n is important in understanding the determinant. Suppose τ is a permutation on n elements. If there exist $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $\tau(i) = j$, $\tau(j) = i$ and $\tau(k) = k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$, we say that τ is a **transposition**.

- (a) Let $\sigma, \tau \in S_n$. Define $\sigma\tau := \sigma \circ \tau$. That is, $\sigma\tau$ is τ composed with σ. Prove that $\sigma\tau \in S_n$.
- (b) Let $\sigma_1, \sigma_2, \sigma_3 \in S_n$. Prove that $(\sigma_1\sigma_2)\sigma_3 = \sigma_1(\sigma_2\sigma_3)$.
- (c) Let $I\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ denote the identity permutation. That is, $I(i) := i$ for all $i \in \{1, \ldots, n\}$. Show that, for all $\sigma \in S_n$, $I\sigma = \sigma I = \sigma$.
- (d) Given $\sigma \in S_n$, show that there exists a unique element $\sigma^{-1} \in S_n$ such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = I$.

We note in passing that properties (a) through (d) show that S_n is a **group**.

- (e) Given any permutation $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, there exists a finite number of transpositions $\tau_1, \ldots, \tau_j$ such that $\sigma = \tau_1\tau_2 \cdots \tau_j$. (Hint: induct on n.)
- (f) Let $\sigma \in S_n$. Suppose $\tau_1, \ldots, \tau_j$ are transpositions and $\sigma = \tau_1 \cdots \tau_j \in S_n$. Define $\mathrm{sign}(\sigma) := (-1)^j$. Show that $\mathrm{sign}(\sigma)$ is well-defined. That is, if we can write $\sigma = \tau_1' \cdots \tau_k'$ for some other transpositions $\tau_1', \ldots, \tau_k'$, then $(-1)^j = (-1)^k$. (Hint: Let σ be a transposition. Suppose $\sigma(i) = j$ and $\sigma(j) = i$ where $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Consider the elementary $n \times n$ matrix T_σ that swaps row i with row j. Recall that

$-1 = \det(T_\sigma)$. Let τ be another transposition. Relate $\det(T_\sigma T_\tau)$ to $\sigma\tau$ using the multiplicative property of the determinant.)

Exercise 8. Let A, B be invertible $n \times n$ matrices. Show that there is a sequence of elementary row operations $E_1, \ldots, E_j$ such that $E_1 \cdots E_j A = B$. (Hint: first try to prove this for $B = I_n$.)