Please provide complete and well-written solutions to the following exercises.

Due May 7, in the discussion section.

Assignment 5

Exercise 1. Section 2.5, Exercise 1(acd) in the textbook.

Exercise 2. Define $\beta := \{(-4,3),(2,-1)\}$ and define $\beta' := \{(2,1),(-4,1)\}$. Find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

Exercise 3. Let A, B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\mathrm{tr}(A) := \sum_{i=1}^{n} A_{ii}.$$

Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(A) = \mathrm{tr}(A^t)$.

Exercise 4. Let A, B be similar $n \times n$ matrices. Show that $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Exercise 5. Let A, B, C be $n \times n$ matrices.

(a) Show that A is similar to A.
(b) Show that, if A is similar to B, then B is similar to A.
(c) Show that, if A is similar to B, and B is similar to C, then A is similar to C.

Combining parts (a),(b) and (c) shows that similarity is an equivalence relation on the space of $n \times n$ matrices.

Exercise 6. Let p be a positive prime integer. Consider the set $\mathbf{Z}/p\mathbf{Z} := \{0, 1, \ldots, p-1\}$. Let x, y be elements of $\mathbf{Z}/p\mathbf{Z}$. We define addition and multiplication on $\mathbf{Z}/p\mathbf{Z}$ by the formulas $x + y := (x + y) \bmod p$, and $x \cdot y := (xy) \bmod p$. Here $(x + y)$ denotes addition in $\mathbf{Z}$, and (xy) denotes multiplication in $\mathbf{Z}$. With these two definitions of addition and multiplication, prove that $\mathbf{Z}/p\mathbf{Z}$ is a field.

Exercise 7. Let V be a finite-dimensional vector space over $\mathbf{R}$. Define V^* to be the set of linear transformations $T \colon V \to \mathbf{R}$. Show that V is isomorphic to V^*.

Exercise 8. Let V be a finite-dimensional vector space over a field $\mathbf{F}$. Suppose $P \colon V \to V$ is a linear transformation such that $P^2 = P$. Such a linear transformation is called a projection. Prove that, for any $v \in V$, there exist unique vectors $n, w \in V$ such that $v = n + w$, where $n \in N(P)$ and $w \in R(P)$.