Please provide complete and well-written solutions to the following exercises.

Due April 16, in the discussion section.

Assignment 2

Exercise 1. Section 1.6, Exercise 1(acdejk) in the textbook.

Exercise 2. Consider the vectors $u_1 = (2, 3, -1), u_2 = (1, 4, -2), u_3 = (-8, 12, -4), u_4 = (1, 37, -17)$ and $u_5 = (-3, -5, 8)$. Then $\text{span}(u_1, u_2, u_3, u_4, u_5) = \mathbf{R}^3$. Find a subset of the vectors $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbf{R}^3$.

Exercise 3. The set of the solutions to the system of linear equations

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 3x_2 + x_3 = 0$$

is a subspace of $\mathbf{R}^3$. Find a basis for this subspace.

Exercise 4. Section 2.1, Exercise 7 in the textbook.

Exercise 5. Section 2.1, Exercise 9 in the textbook.

Exercise 6. Section 2.1, Exercise 21 in the textbook.

Exercise 7. Let $C(\mathbf{R})$ denote the vector space over $\mathbf{R}$ of all continuous functions on $\mathbf{R}$. Determine which of the following subsets of $C(\mathbf{R})$ are subspaces of $C(\mathbf{R})$.

(1) $P(\mathbf{R})$, the set of all polynomials of one real variable.
(2) The set of all $f \in C(\mathbf{R})$ such that $f(1/2)$ is a rational number.
(3) The set of all $f \in C(\mathbf{R})$ such that $\int_0^1 f(t)dt = 0$.
(4) The set of all $f \in C(\mathbf{R})$ such that df/dt exists and such that $df/dt + 2f = 0$.

Exercise 8. Is the union of two subspaces always a subspace? Explain.

Exercise 9. Prove that $P(\mathbf{R})$ is infinite-dimensional.

Exercise 10. Let V be a vector space over a field $\mathbf{F}$. Suppose V has a finite spanning set. Prove that V has a basis.

Exercise 11. What are all possible dimensions of all possible subspaces of $\mathbf{R}^3$? Give examples for all such dimensions.

Exercise 12. Let V be a vector space over a field $\mathbf{F}$. Let $S, T \subseteq V$ be subspaces. Define $S + T := \{s + t \colon s \in S, t \in T\}$.

(a) Prove that $S + T$ is a subspace of V.
(b) Prove that $S + T$ is the smallest subspace containing $S \cup T$. That is, if $W \subseteq V$ is a subspace such that $S \subseteq W$ and $T \subseteq W$, then $S + T \subseteq W$.

(c) Prove: $\dim(S) + \dim(T) = \dim(S \cap T) + \dim(S + T)$.

(Hint for (c): In the finite-dimensional case, start with a basis $\{u_1, \ldots, u_n\}$ for $S \cap T$. Then, add more vectors to $\{u_1, \ldots, u_n\}$ to get a basis for S. Then, add more vectors to $\{u_1, \ldots, u_n\}$ to get a basis for T. What can you say about the union of all of these vectors?)