Please provide complete and well-written solutions to the following exercises.

Due April 9, in the discussion section.

Assignment 1

Exercise 1. Section 1.2, Exercise 1(abghk) in the textbook.

Exercise 2. Let V be a vector space over a field $\mathbf{F}$. Using the axiomatic definitions of fields and vector spaces, prove the following facts:

- $\forall\, v \in V$, $0 \cdot v = 0$.
- $\forall\, v \in V$, $(-1) \cdot v = -v$.
- $\forall\, \alpha \in F$, and for $0 \in V$, $\alpha \cdot 0 = 0$.
- $\forall\, \alpha \in \mathbf{F}$, $\forall\, v \in V$, $\alpha \cdot (-v) = (-\alpha) \cdot v = -(\alpha \cdot v)$.

Exercise 3. Section 1.3, Exercise 8(abf) in the textbook.

Exercise 4. Show that the intersection of two subspace is also a subspace.

Exercise 5. Section 1.4, Exercise 3(c) in the textbook.

Exercise 6. Section 1.5, Exercise 1(abdef) in the textbook.

Exercise 7. Let V be a vector space over a field $\mathbf{F}$. Let $\{u_1, \ldots, u_n\} \subseteq V$ satisfy the following property. For any $u \in V$, there exist unique scalars $\alpha_1, \ldots, \alpha_n \in \mathbf{F}$ such that

$$u = \alpha_1 u_1 + \cdots + \alpha_n u_n.$$

Prove that $\{u_1, \ldots, u_n\}$ is a basis of V.

Exercise 8. Give an example of a subset of $\mathbf{R}^2$ that is closed under vector addition, but which is not closed under multiplication by scalars.

Exercise 9. Give an example of a subset of $\mathbf{R}^2$ that is closed under scalar multiplication, but which is not closed under vector addition.

Exercise 10. Find three nonzero, distinct vectors $f, g, h \in \mathbf{R}^3$ satisfying the following properties: $\mathrm{span}(f, g) = \mathrm{span}(g, h) = \mathrm{span}(f, g, h)$, and $\mathrm{span}(f, h) \neq \mathrm{span}(f, g, h)$.

Exercise 11. Consider the subset of the integers $X = \{0, 1, 2, \ldots, 18, 19\}$. For any $x, y \in X$, define the addition operation $x + y := (x + y) \bmod 20$, where $(x + y)$ denotes addition in $\mathbf{Z}$. For any $x, y \in X$, define the multiplication operation $x \cdot y := (xy) \bmod 20$, where (xy) denotes multiplication in $\mathbf{Z}$. Note that X is now closed under multiplication and addition with these definitions. (For example, $14 + 12 = 6 \bmod 20$, $13 \cdot 7 = 11 \bmod 20$.) Is X a field? Prove your assertion.

Exercise 12. Consider the set $F = \{a + b\sqrt{2} \colon a, b \in \mathbf{Q}\}$. Prove that F is a field.