115A Final Practice Solutions

These are solutions to the practice midterm here: `http://math.berkeley.edu/sites/default/files/pages/F03_Final_Exam-K.Ribet_.pdf`

1. Question 1

Let A be an $n \times n$ matrix. Assume there exists a nonzero row vector y such that $yA = y$. Prove there exists a nonzero column vector x such that $Ax = x$.

Solution. $yA - y = 0$, i.e. $y(A - I_n) = 0$. That is, there exists a linear combination of the rows of $A - I_n$ that yields the zero vector. From a Corollary from the notes (Corollary 3.15 from the third set of notes), we conclude that $A - I_n$ has rank less than n. (Since the rank of $A - I_n$ is equal to the dimension of the span of its rows.) From the rank-nullity theorem, $A - I_n$ has a nonzero vector in its null space. That is, there exists a nonzero vector x such that $(A - I_n)x = 0$. That is, there exists a nonzero vector x such that $Ax = x$, as desired.

2. Question 2

Let A, B be $n \times n$ matrix over a field $\mathbf{F}$. Assume $A^2 = A$ and $B^2 = B$. Prove that A and B are similar if and only if they have the same rank.

Solution. Suppose A and B are similar. That is, there exists an invertible matrix Q such that $A = QBQ^{-1}$. Since invertible matrices preserve the rank (Lemma 3.10 from the third set of notes), we conclude that $\mathrm{rank}(A) = \mathrm{rank}(B)$, as desired.

Now, suppose A and B have the same rank. We need to find an invertible matrix Q such that $A = Q^{-1}BQ$. From Exercise 8 of homework 5, if $v \in \mathbf{F}^n$, then there exist unique vectors n, w and n', w' such that $n \in N(L_A)$, $w \in R(L_A)$, $n' \in N(L_B)$, $w' \in R(L_B)$ and such that $v = n + w$ and $v = n' + w'$. (Note that since $w \in R(L_A)$, there exists $z \in \mathbf{F}^n$ such that $Az = w$, so using $A^2 = A$, we have $A^2 z = Aw = Az = w$, so $Aw = w$. Similarly, $Bw' = w'$.)

Since A and B have the same rank, their null spaces have the same dimension k, and their ranges have the same dimension $n - k$, for some $0 \leq k \leq n$. So, let $(a_1, \ldots, a_k)$ be a basis for $N(L_A)$, let $(b_1, \ldots, b_k)$ be a basis for $N(L_B)$, let $(a_{k+1}, \ldots, a_n)$ be a basis for $R(L_A)$ and let $(b_{k+1}, \ldots, b_n)$ be a basis for $R(L_B)$. By the uniqueness property discussed above (and Exercise 7 on homework 1), we conclude that $\alpha = (a_1, \ldots, a_n)$ is a basis of $\mathbf{F}^n$ and $\beta = (b_1, \ldots, b_n)$ is a basis of $\mathbf{F}^n$. So, let T be a linear transformation such that $T(a_i) = b_i$ for all $1 \leq i \leq n$. (Such a T exists by Theorem 4.4 in the second set of notes.) Note that $[T]_\alpha^\beta$ is the identity matrix by the definition of T, so T is invertible (by Corollary 6.11 in the second set of notes). So, if γ is the standard basis of $\mathbf{F}^n$, and if we define $Q = [T]_\gamma^\gamma$ as the matrix representation of T in the standard basis, then, for any $v \in \mathbf{F}^n$, when we write $n + w = v = \sum_{i=1}^n \alpha_i a_i$ with $\alpha_i \in \mathbf{F}$ for all $1 \leq i \leq n$, we get

$$Av = A(\sum_{i=1}^n \alpha_i a_i) = A(\sum_{i=k+1}^n \alpha_i a_i) = \sum_{i=k+1}^n \alpha_i a_i$$

(recall $Aa_i = a_i$ for all $k + 1 \leq i \leq n$, and $Bb_i = b_i$ for all $k + 1 \leq i \leq n$), so

$$Q^{-1}BQv = Q^{-1}BQ(\sum_{i=1}^n \alpha_i a_i) = Q^{-1}B(\sum_{i=1}^n \alpha_i b_i) = Q^{-1}(\sum_{i=k+1}^n \alpha_i b_i) = \sum_{i=k+1}^n \alpha_i a_i.$$

That is, $A = Q^{-1}BQ$, as desired.

3. Question 3

Suppose $T\colon V \to V$ is a linear transformation on a finite-dimensional inner product space. Let T^* be the adjoint of T. Show that every $v \in V$ can be written uniquely as $v = a + b$ where $a \in N(T)$ and $b \in R(T^*)$.

Solution 1. Define $W := N(T)^\perp$. We claim that $R(T^*) = N(T)^\perp$. To see this, let $x \in R(T^*)$. Then, there exists $z \in V$ such that $T^*z = x$. Note that, for any $y \in N(T)$, we have $\langle x, y\rangle = \langle T^*z, y\rangle = \langle z, Ty\rangle = 0$. That is, $x \in N(T)^\perp$. Therefore, $R(T^*) \subseteq N(T)^\perp$. From the dimension theorem for orthogonal complements (Corollary 4.18 in the fifth set of notes), $\dim(N(T)) + \dim(N(T)^\perp) = \dim(V)$. From the rank-nullity theorem, $\dim(N(T)) + \dim(R(T)) = \dim(V)$. Combining these two equalities, $\dim(N(T)^\perp) = \dim(R(T))$. Using, e.g. the matrix representation of T, Theorem 5.12 in the fifth set of notes, and Exercise 5.16 in the fifth set of notes, we know that $\dim(R(T)) = \dim(R(T^*))$. In conclusion, $\dim(N(T)^\perp) = \dim(R(T^*))$. Since $R(T^*) \subseteq N(T)^\perp$, and both are subspaces of equal dimension, we conclude that in fact $R(T^*) = N(T)^\perp$ (by Theorem 7.1 in the first set of notes).

In conclusion, $R(T^*) = N(T)^\perp$. So, by Corollary 4.9 in the fifth set of notes, every vector $v \in V$ can be written uniquely as $v = a + b$ where $a \in N(T)$ and $b \in N(T)^\perp = R(T^*)$.

Solution 2. Define $W := N(T)^\perp$. We claim that $R(T^*) = N(T)^\perp$. As a preliminary claim, we show that $N(T^*) = R(T)^\perp$. To see this, note that $x \in N(T^*)$ if and only if $T^*x = 0$, if and only if $\langle T^*x, y\rangle = 0$ for all $y \in V$ (by the Riesz Representation Theorem, Theorem 5.6 in the fifth set of notes), if and only if $\langle x, Ty\rangle = 0$ for all $y \in V$, if and only if $x \in R(T)^\perp$. So, $N(T^*) = R(T)^\perp$. Now, taking the orthogonal complement of both sides, we get $N(T^*)^\perp = R(T)$. Then, using T^* in place of T and noting that $T^{**} = T$, we get $N(T)^\perp = R(T^*)$, as desired.

In conclusion, $R(T^*) = N(T)^\perp$. So, by Corollary 4.9 in the fifth set of notes, every vector $v \in V$ can be written uniquely as $v = a + b$ where $a \in N(T)$ and $b \in N(T)^\perp = R(T^*)$.

4. Question 4

Let A be a symmetric real matrix such that $\mathrm{Tr}(A^2) = 0$. Show that $A = 0$.

Solution. Suppose A has entries a_{ij} where $1 \le i, j \le n$. Let $1 \le i \le n$, then entry (i, i) of the matrix A^2 is $\sum_{j=1}^n a_{ij}a_{ji}$, by the definition of matrix multiplication. Since A is symmetric, we have $a_{ij}a_{ji} = a_{ij}^2$. In conclusion,

$$0 = \mathrm{Tr}(A) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}a_{ji} = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2.$$

A sum of squared real numbers can only be zero if all of the real numbers are zero. That is, we must have $a_{ij} = 0$ for all $1 \le i, j \le n$.

5. Question 5

Let $T\colon V \to W$ be a linear transformation between finite-dimensional vector spaces. Let X be a subspace of W. Let $T^{-1}(X)$ be the set of vectors in V that map to X. Show that $T^{-1}(X)$ is a subspace of V and that $\dim T^{-1}(X) \ge \dim V - \dim W + \dim X$. (I found this question to be pretty hard myself.)

Solution 1. Assume for now that $T^{-1}(X)$ is a subspace of V. (We verify this in Solution 2.) Now, note that $T\colon T^{-1}(X) \to X$ is a linear transformation. So, by the rank-nullity theorem, we have $\dim(T^{-1}(X)) = \dim(X \cap R(T)) + \text{nullity}(T)$. Here $R(T)$ denotes the range of $T\colon V \to W$, and $\text{nullity}(T)$ is the nullity of $T\colon V \to W$. (Note that, since X is a subspace, $0 \in X$, so $T^{-1}X$ contains the null space of $T\colon V \to W$, so the null space of $T\colon T^{-1}(X) \to X$ is equal to the null space of $T\colon V \to W$.) From Exercise 12 on Homework 2, $\dim(X \cap R(T)) = \dim(X) + \dim(R(T)) - \dim(X + R(T))$. Since $X + R(T) \subseteq W$, we have $\dim(X + R(T)) \leq \dim(W)$, so $-\dim(X + R(T)) \geq -\dim(W)$. Putting everything together,

$$\begin{aligned}\dim(T^{-1}(X)) &= \dim(X) + \text{nullity}(T) + \dim(R(T)) - \dim(X + R(T))\\ &\geq \dim(X) + \text{nullity}(T) + \dim(R(T)) - \dim(W).\end{aligned}$$

Finally, using the rank-nullity theorem again, $\text{nullity}(T) + \dim(R(T)) = \dim(V)$. That is,

$$\dim(T^{-1}(X)) \geq \dim(X) + \dim(V) - \dim(W)$$

Solution 2. We first verify that $T^{-1}(X)$ is a subspace of V. Let $a, b \in T^{-1}(X)$, and let $\alpha \in \mathbf{F}$. Then there exist $x, y \in X$ such that $T(a) = x$ and $T(b) = y$. Since T is linear, we have $T(a+b) = x + y$. Since X is a subspace of W, we have $x + y \in X$. So, by definition of $T^{-1}(X)$, we know that $a+b \in T^{-1}(X)$. That is, $T^{-1}(X)$ is closed under addition. Now, since T is linear, we have $T(\alpha a) = \alpha T(a) = \alpha x$. Since X is a subspace of W, we have $\alpha x \in X$. So, by definition of $T^{-1}(X)$, we have $\alpha a \in T^{-1}(X)$. That is, $T^{-1}(X)$ is closed under scalar multiplication. In summary, $T^{-1}(X)$ is a subspace of V.

For the dimension calculation, we find it easier to prove this assertion for matrices. Using coordinate representations and isomorphisms as necessary, it suffices to prove: if A is an $n \times m$ matrix, and if X is the subspace of $\mathbf{F}^n$ consisting of all vectors that are zero in the first k entries, then $L_A\colon \mathbf{F}^m \to \mathbf{F}^n$ satisfies $\dim(L_A^{-1}(X)) \geq m - n + \dim(X)$. (Since all vector spaces of fixed dimension are isomorphic, for the purposes of this problem, we can use our favorite subspace X, as just described.)

Let s be equal to the dimension of the span of the first k rows of A. Note that $s \leq k$ by the definition of s. Also, by the definition of X, we know that $L_A^{-1}(X)$ is the set of vectors v in $\mathbf{F}^m$ such that the first k entries of Av are zero. That is, (if we use the standard inner product), $L_A^{-1}(X)$ is the set of vectors in $\mathbf{F}^m$ which are perpendicular to the first k rows of A. That is, $L_A^{-1}(X)$ is the orthogonal complement of the span of the first k rows of A. By the definition of s, and the dimension theorem for complemented subspaces (Corollary 4.18 in the fifth set of notes), we therefore have $\dim(L_A^{-1}(X)) = m - s$. We are required to show that $\dim L_A^{-1}(X) \geq \dim \mathbf{F}^m - \dim \mathbf{F}^n + \dim X$. That is, we are required to show that $m - s \geq m - n + (n - k)$. Equivalently, we are required to show that $-s \geq -k$. Equivalently, we are required to show that $s \leq k$. But we already showed that $s \leq k$, so we are done.

6. Question 6

Suppose V is a real finite-dimensional inner product space and $T\colon V \to V$ is a linear transformation such that $\langle T(x), T(y)\rangle = 0$ whenever $x, y \in V$ satisfy $\langle x, y\rangle = 0$. Assume there is a nonzero $v \in V$ such that $\|T(v)\| = \|v\|$. Show that T is orthogonal. (I thought this problem was fairly difficult.)

Solution 1. Let $x \in V$ with $x \neq 0$ such that $\langle x, v\rangle = 0$. Define $\lambda := \|v\| / \|x\|$ and let $y := \lambda x$. Then $\|y\| = \|v\|$ and $\langle y, v\rangle = 0$. So, $\langle v + y, v - y\rangle = \|v\|^2 - \|y\|^2 = 0$. So, by assumption on T, we have $0 = \langle T(v+y), T(v-y)\rangle = \|Tv\|^2 - \|Ty\|^2$, so $\|y\|^2 = \|v\|^2 = \|Tv\|^2 = \|Ty\|^2$. Using the definition of y, we therefore have $\|Tx\|^2 = |\lambda|^2 \|Ty\|^2 = |\lambda|^2 \|y\|^2 = \|x\|^2$. That is, $\|Tx\|^2 = \|x\|^2$ for all $x \in V$ with $\langle x, v\rangle = 0$. Since any $x \in V$ can be written uniquely as $x = n + w$, where n is perpendicular to v, and w is in the span of v, we conclude that $\|Tx\|^2 = \|T(n) + T(w)\|^2 = \langle T(n)+T(w), T(n)+T(w)\rangle = \|Tn\|^2 + \|Tw\|^2 = \|n\|^2 + \|w\|^2 = \|n + w\|^2 = \|x\|^2$. Here we used the assumption on T, and the Pythagorean Theorem. That is, T is an orthogonal transformation, since $\|Tx\|^2 = \|x\|^2$ for all $x \in V$.

Solution 2. Let $v = v_1$ be the nonzero vector such that $\|Tv\| = \|v\|$. Without loss of generality, $\|v\| = 1$. This vector can be completed to a finite basis of V by Corollary 6.14(f) in the first set of notes. Then, using Gram-Schmidt orthogonalization on this basis (starting with v_1 which already has norm 1), we may assume that there exists $\beta = (v_1, \dots, v_n)$ an orthonormal basis of V such that $v = v_1$.

Now, define a linear transformation $R\colon V \to V$ such that $R(v_i) = v_i$ for all $3 \leq i \leq n$, $R(v_1) = (v_1 + v_2)/\sqrt{2}$, and $R(v_2) = (v_2 - v_1)/\sqrt{2}$. Note that the matrix representation $[R]_\beta^\beta$ is a matrix with ones on the diagonal entries (i, i) for all $3 \leq i \leq n$, the upper left corner of the matrix is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, and the remaining entries of $[R]_\beta^\beta$ are zero. Since the 2×2 matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is orthogonal, it follows that the matrix $[R]_\beta^\beta$ is orthogonal. From Proposition 8.6 in the fifth set of notes, R is therefore orthogonal. That is, $\langle Rv_1, Rv_2\rangle = \langle v_1, v_2\rangle = 0$. However, by the definition of T, we have

$$\begin{aligned} 0 &= 2\langle Rv_1, Rv_2\rangle = 2\langle TRv_1, TRv_2\rangle = \langle T(v_1 + v_2), T(v_2 - v_1)\rangle \\ &= \langle Tv_1, Tv_2\rangle - \langle Tv_1, Tv_2\rangle + \langle Tv_2, Tv_2\rangle - \langle Tv_1, Tv_1\rangle = \langle Tv_2, Tv_2\rangle - \langle Tv_1, Tv_1\rangle. \end{aligned}$$

That is,

$$\langle Tv_2, Tv_2\rangle = \langle Tv_1, Tv_1\rangle.$$

By assumption of v_1, we have $\langle Tv_1, Tv_1\rangle = \|v_1\|^2 = 1$. Therefore, $\langle Tv_2, Tv_2\rangle = 1$.

Similarly, given any $2 \leq j \leq n$, we can define a linear transformation $R\colon V \to V$ such that $R(v_i) = v_i$ for all $2 \leq i \leq n$ with $i \neq j$, $R(v_1) = (v_1 + v_j)/\sqrt{2}$, and $R(v_j) = (v_j - v_1)/\sqrt{2}$. We similarly conclude that $\langle Tv_j, Tv_j\rangle = 1$. That is, for all $1 \leq j \leq n$, we have $\langle Tv_j, Tv_j\rangle = 1$.

Now, given any $x, y \in V$, we express x, y in the orthonormal basis β as $x = \sum_{i=1}^n \alpha_i v_i$ and $y = \sum_{i=1}^n \gamma_i v_i$ where $\alpha_i, \gamma_i \in \mathbf{F}$ for all $1 \leq i \leq n$. Note that, by our assumption on T, we have $\langle Tv_i, Tv_j\rangle = 0$ for all $i \neq j$, since $\langle v_i, v_j\rangle = 0$ when $i \neq j$. Therefore,

$$\langle Tx, Ty\rangle = \langle \sum_{i=1}^n \alpha_i Tv_i, \sum_{i=1}^n \gamma_i Tv_i\rangle = \sum_{i=1}^n \alpha_i\gamma_i \langle Tv_i, Tv_i\rangle = \sum_{i=1}^n \alpha_i\gamma_i = \langle x, y\rangle.$$

That is, T is an orthogonal transformation. (Since $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in V$, we know that $\langle T^*Tx, y\rangle = \langle x, y\rangle$ for all $x, y \in V$, so $T^*Tx = x$ for all $x \in V$, by the Riesz Representation Theorem (Theorem 5.6 in the fifth set of notes). So, $T^*T = I_V$. Also, since $\|Tx\| = \|x\|$ for all $x \in V$, we know that $T\colon V \to V$ is one-to-one, so $T\colon V \to V$ is invertible. Since $T^*T = I_V$, we know that T has unique inverse T^*, so that $TT^* = I_V$ as well.)

7. Question 7

(This question is outside our course material.)

8. Question 8

Let $\mathbf{F}$ be a finite field with $q \geq 2$ elements. Let V be an n-dimensional vector space. In terms of n and q, compute the number of 1-dimensional subspaces of V and the number of linear transformations $V \to V$ that have rank 1. (This question is pretty lengthy for an exam, I think.)

Solution. Without loss of generality, we assume that $V = \mathbf{F}^n$. Let W be any one-dimensional subspace of V. Then, there exists a nonzero vector $v \in V$ such that W is the span of v. Since v is nonzero, v has some nonzero entry $\alpha \in \mathbf{F}$ where $\alpha \neq 0$. Consider the map $T\colon \mathbf{F} \to \mathbf{F}$ defined by $T(\beta) = \alpha\beta$ for all $\beta \in \mathbf{F}$. Since $\mathbf{F}$ is a field, the function T is one-to-one and onto. Since $\mathbf{F}$ has q elements, $\mathbf{F}$ has $q-1$ distinct nonzero elements. That is, there exist exactly $q-1$ distinct nonzero elements in the set $\{\alpha\beta \colon \beta \in \mathbf{F}\}$. Therefore, there exist at least $q-1$ distinct nonzero elements in the span of v, namely $\{\alpha v \colon \alpha \in \mathbf{F}, \alpha \neq 0\}$. We claim that the span of v has exactly $q-1$ distinct nonzero elements. We can find $q-1$ distinct nonzero elements, as specified above, from the set $\{\alpha v \colon \alpha \in \mathbf{F}, \alpha \neq 0\}$. However, any other element in the span of v must be of this form, by the definition of span. Therefore, there exist exactly $q-1$ nonzero elements in the span of v.

Now, let $v, w \in V$. We claim that either $\mathrm{span}(v) = \mathrm{span}(w)$ or $\mathrm{span}(v) \cap \mathrm{span}(w) = \{0\}$. To prove this, suppose $\mathrm{span}(v) \cap \mathrm{span}(w)$ has some nonzero element. We will then prove that $\mathrm{span}(v) = \mathrm{span}(w)$. Let $z \in \mathrm{span}(v) \cap \mathrm{span}(w)$ with $z \neq 0$. Then $z \in \mathrm{span}(v)$ and $z \in \mathrm{span}(w)$. Then there exist nonzero scalars $\alpha, \beta \in \mathbf{F}$ such that $z = \alpha v$ and $z = \beta w$. That is, $v = \alpha^{-1}\beta w$. So, by the definition of span we have $\mathrm{span}(v) = \mathrm{span}(w)$, as desired.

Okay, so we can use this result to count the number of 1-dimensional subspaces of V. Consider the set of all 1-dimensional subspaces of V. This set is equal to the set of all spans of v, where v ranges over all $v \in V$. By the previous result, every nonzero vector in V belongs to exactly one set of the form $\mathrm{span}(v')$, for some $v' \in V$. Also, there exist exactly $q-1$ nonzero elements in $\mathrm{span}(v')$ for any fixed $v' \in V$. Since $V = \mathbf{F}^n$ and $\mathbf{F}$ has exactly q elements, we know that V has exactly $q^n - 1$ nonzero elements. In summary, there are exactly $(q^n-1)/(q-1)$, 1-dimensional subspaces in V.

Finally, note that any rank 1 linear transformation on V can be obtained by first projecting onto a 1-dimensional subspace, and then multiplying by any nonzero vector. Specifically, let $T\colon V \to V$ have rank 1. Note that $N(T)$ has dimension $n-1$ by the rank-nullity theorem, so $N(T)^{\perp}$ has dimension 1, by Corollary 4.18 in the fifth set of notes. By Corollary 4.19 in the fifth set of notes, every $v \in V$ can be written uniquely as $v = n + y$ where $n \in N(T)$ and $y \in N(T)^{\perp}$. Specifically, if $P\colon V \to V$ denotes projection onto $N(T)^{\perp}$, then $v = (v - Pv) + Pv$. (In the proof of Corollary 4.19, we defined $n = v - Pv$ and $y = Pv$.) Let w be a nonzero vector in $N(T)^{\perp}$. Then we can express $P\colon V \to V$ as an orthogonal projection onto w. That is, $P(v) = w\langle v, w\rangle/\langle w, w\rangle$. And any v can be written as $v = (v - Pv) + Pv$, so that $v - Pv \in N(T)$, so

$$Tv = T((v - Pv) + Pv) = TPv = T(w)\langle v, w\rangle/\langle w, w\rangle.$$

That is, we wrote T as a projection onto the span of w, multiplied by another vector $T(w)$. Conversely, given any projection $P\colon V \to V$ onto a one-dimensional subspace of V, and given any $z \in V$ with $z \neq 0$, the function $T\colon V \to V$ defined by $T(v) = z\langle v, w\rangle/\langle w, w\rangle$ is a rank 1 linear transformation (where w is any nonzero vector in $R(P)$). In summary, the number of rank 1 projections from V to V is bounded by the number of one-dimensional subspaces of V, multiplied by the number of nonzero elements of V. In fact, every distinct one-dimensional subspace and every distinct nonzero $z \in V$ yield a rank 1 linear transformation. So, the number of rank 1 linear transformations on V is equal to: the number of one-dimensional subspaces of V, multiplied by the number of nonzero elements of V. That is, the number of rank 1 linear transformations on V is equal to $(q^n - 1)^2/(q - 1)$.