115A Midterm 2 Solutions[1]

1. Question 1

True/False

(a) For any $n \geq 1$, every linear transformation on an n-dimensional vector space has n distinct eigenvalues.

FALSE: The 2×2 identity matrix has all eigenvalues 1.

(b) Any linear transformation on a 4-dimensional vector space that has fewer than 4 distinct eigenvalues is not diagonalizable.

FALSE: The 4×4 identity matrix has all eigenvalues 1, but it is diagonal, so it is diagonalizable.

(c) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

FALSE: The standard basis of $\mathbf{R}^n$ gives an orthonormal set of eigenvectors for the identity matrix, and all of these vectors have eigenvalue 1.

(d) Let V be a vector space over $\mathbf{C}$. Let $T\colon V \to V$ be a linear transformation. If 0 is the only eigenvalue of T, then $T = 0$.

FALSE: Recall that a 2×2 matrix has at most 2 eigenvalues by the Fundamental Theorem of Algebra. Now, the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has exactly two zero eigenvalues, but it is not zero.

(e) Let A, B be 3×3 matrices such that $AB = -BA$. Then either A or B is non-invertible.

TRUE: Taking the determinant of the identity $AB = -BA$ and applying the multiplicative property of the determinant, we get $\det(A)\det(B) = -\det(B)\det(A)$. If both A,B are invertible, then they have nonzero determinants, so we get $1 = -1$, a contradiction. We conclude that at least one of A, B has zero determinant. That is, at least one of A, B is not invertible.

2. Question 2

Let V be a (possibly infinite-dimensional) vector space over a field $\mathbf{F}$. Suppose $P\colon V \to V$ is a linear transformation such that $P^2 = P$. Such a linear transformation is called a projection. Prove that, for any $v \in V$, there exist unique vectors $n, w \in V$ such that $v = n + w$, where $n \in N(P)$ and $w \in R(P)$.

Solution. Write $v = (v - P(v)) + P(v)$. Define $n := v - P(v)$ and define $w := P(v)$. Then $v = n+w$. Also, $n \in N(P)$ since $P(v-P(v)) = P(v)-P(P(v)) = P(v)-P(v) = 0$, using the assumption $P^2 = P$. Lastly, $w \in R(P)$, by definition of w (that is, using $w = P(v)$). We have therefore proven existence. We now prove uniqueness. Suppose $v = n' + w'$ where $n' \in N(P)$ and $w' \in R(P)$. Then $n + w = n' + w'$, so $n - n' = w' - w$. So, if we define $x = n - n'$, then $x \in N(P)$ and $x \in R(P)$. Since $x \in R(P)$, there exists $z \in V$ such that $P(z) = x$. Then $P^2(z) = P(x) = 0$, using $x \in N(P)$, Since $P^2 = P$, we have $P(z) = P(x) = 0$, so that $z \in N(P)$. Since $P(z) = 0$ and $P(z) = x$, we have $x = 0$. That is, $0 = n - n' = w' - w$, so that $n = n'$ and $w = w'$. That is, n, w are unique; there is only one way to write $v = n + w$ where $n \in N(P)$ and $w \in R(P)$.

3. Question 3

In the following questions, you do not necessarily need to diagonalize the matrix to answer the question.

(i) Is the following matrix diagonalizable over $\mathbf{R}$? Why or why not?

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}.$$

Solution. This matrix is diagonalizable over $\mathbf{R}$. Its characteristic polynomial is $(2-\lambda)(3-\lambda)(4-\lambda)$, which is a product of distinct real roots. We can therefore diagonalize A over $\mathbf{R}$. This was Corollary 5.3 in the fourth set of notes.

(ii) Is the following matrix diagonalizable over $\mathbf{R}$? Why or why not?

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 1 & 3 \end{pmatrix}.$$

This matrix is also diagonalizable over $\mathbf{R}$, because its characteristic polynomial is a product of distinct real roots, so Corollary 5.3 applies as before. Observe, the characteristic polynomial is

$$(2-\lambda)[(3-\lambda)^2 - 4] = (2-\lambda)(\lambda^2 - 6\lambda + 5) = (2-\lambda)(\lambda - 5)(\lambda - 1).$$

Therefore, the characteristic polynomial of A has three distinct real roots.

(iii) Is the following matrix diagonalizable over $\mathbf{C}$? Why or why not?

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This matrix is diagonalizable over $\mathbf{C}$. Its characteristic polynomial is $\lambda^2 + 1 = (\lambda + \sqrt{-1})(\lambda - \sqrt{-1})$, which is a product of distinct complex roots. We can therefore diagonalize A over $\mathbf{C}$ by as Corollary 5.3 in the fourth set of notes.

4. Question 4

Let A be the real matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

Find bases for the null space $N(L_A)$ and for the range $R(L_A)$. Then, find a diagonal 3×3 matrix D whose entries are zero or one such that there exist invertible 3×3 matrices P, Q with $D = QAP^{-1}$. (You may assume that such D, Q, P exist, you do not need to find P, and you do not need to find Q.)

Solution: Since A is 3×3, recall that $L_A \colon \mathbf{R}^3 \to \mathbf{R}^3$. Let's do some row reductions.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

So, A must have rank 2 (since elementary row operations are invertible, and invertible transformations preserve the rank of a matrix), so the null space is 1-dimensional, and the range space is 2-dimensional, by the rank-nullity theorem. The null space is unchanged by invertible transformations, so a basis of the null space of the row-reduced form of A is a basis for the null space of A. So, a basis for the null space of A is

$$\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$$

There are a few ways to find a basis for the range of A. For one method, note that the first and second columns of the row-echelon form of A are part of a basis of $\mathbf{R}^3$. So, the first and second columns of A must be a basis for the range of A, since the row-echelon form was obtained by applying invertible matrices to A. That is, the vectors

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\3\\4\end{pmatrix}\right\}$$

form a basis for the range of A. Actually, any two linearly independent columns of A form a basis for the range of A, since this range is two-dimensional.

Now, to find D, note that since P, Q are invertible, if D exists as stated, then D must have rank 2. Since D is diagonal with zero or one entries, D may therefore have the form

$$D = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$

Actually, D could be any diagonal matrix with two ones and one zero on the diagonal.

5. Question 5

Let n be a positive integer. Let $T\colon P_n(\mathbf{R}) \to \mathbf{R}^{n+1}$ be defined by

$$T(f) := (f(0), f(1), f(2), \ldots, f(n)).$$

(Recall that $P_n(\mathbf{R})$ is the set of all polynomials of degree at most n in a real variable x.) For example, if $n = 3$, then $T(x^2) = (0, 1, 4, 9)$.

Prove that T is linear. Then, prove that T is an isomorphism.

Solution. Let $f, g \in P_n(\mathbf{R})$ and let $\alpha \in \mathbf{R}$. Then

$$\begin{aligned}T(\alpha f + g) &= ((\alpha f + g)(0), \ldots, (\alpha f + g)(n)) = (\alpha f(0) + g(0), \ldots, \alpha f(0) + g(n))\\ &= \alpha(f(0), \ldots, f(n)) + (g(0), \ldots, g(n)) = \alpha T(f) + T(g).\end{aligned}$$

Therefore, T is linear. We now show that T is an isomorphism. From Lemma 6.6 in the second set of notes, it suffices to show that T is both one-to-one and onto. Both properties actually follow from Lagrange Interpolation (Theorem 7.2 in the first set of notes). Given any numbers $y_0, \ldots, y_n$, there exists a unique polynomial $f \in P_n(\mathbf{R})$ such that $f(i) = y_i$ for all $0 \leq i \leq n$. The existence of f implies that T is onto, and the uniqueness of f implies that T is one-to-one. (If $T(f) = T(g)$, then f and g both satisfy $f(i) = g(i) = y_i$ for all $0 \leq i \leq n$, so by the uniqueness part of Lagrange Interpolation, we have $f = g$, so that T is one-to-one). In conclusion, T is both one-to-one and onto, so it is an isomorphism by Theorem 7.2 in the first set of notes.