115A Midterm 2 Practice Solutions

These are solutions to the practice midterm here: `http://math.berkeley.edu/sites/default/files/pages/F03_Second_Midterm-A.Liu_.pdf`

This is an 80 minute exam, so it has a lot more on it than our 50 minute exam.

1. Question 1

True/False

(a) Let V, W be finite dimensional vector spaces over $\mathbf{R}$. Then $\mathcal{L}(V, W)$ and $\mathcal{L}(W, V)$ are isomorphic.

TRUE: Let $T\colon V \to W$. Let α be an ordered basis for V and let β be an ordered basis for W. Suppose V is n-dimensional and W is m dimensional. Define $F(T) := [T]_\alpha^\beta$, $F\colon \mathcal{L}(V, W) \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$. Note that F is an isomorphism. (F is linear, $F(T) = 0$ implies $T = 0$, and given any $m \times n$ real matrix A, there exists $T\colon V \to W$ such that $F(T) = A$, by Theorem 4.4 in the second set of notes.) It therefore suffices to show that, for any positive integers n, m, we have $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$ is isomorphic to $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$. That is, it suffices to show that the set of $m\times n$ real matrices is isomorphic to the set of $n\times m$ real matrices. To show this, let A be an $m\times n$ real matrix, and define $T(A) := A^t$. Then T is an isomorphism, completing the theorem. To see this, note that T is linear, T is one-to-one (since if $T(A) = 0_{n\times m}$, then $A = 0_{m\times n}$), and T is onto (since, if we are given B an $n\times m$ matrix, then $T(B^t) = (B^t)^t = B$).

(b) Let $T\colon V \to V$ be linear. Then the null space $N(T)$ is always contained in the range of T.

FALSE: Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x, y) = (0, y)$. Then T is linear, $N(T)$ is the span of the vector $(1, 0)$, but $R(T)$ is the span of the vector $(0, 1)$. If we had $N(T) \subseteq R(T)$, then $R(T)$ would also contain the span of $(1, 0)$, so that $R(T)$ would be two-dimensional. However, $R(T)$ is just one-dimensional.

(c) There can be no onto linear transformation from $\mathbf{R}^{10}$ to $P_{10}(\mathbf{R})$.

TRUE: Suppose $T\colon \mathbf{R}^{10} \to P_{10}(\mathbf{R})$ is any linear transformation. Recall that $\dim(P_{10}(\mathbf{R})) = 11$ and $\dim(\mathbf{R}^{10}) = 10$. By the rank-nullity theorem, $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\mathbf{R}^{10}) = 10$. So, $\mathrm{rank}(T) = 10 < 11 = \dim(P_{10}(\mathbf{R}))$. Since $R(T)$ is a subspace of $P_{10}(\mathbf{R})$ of dimension less than 11, there must exist some nonzero vector $v \in P_{10}(\mathbf{R})$ such that $v \notin R(T)$. That is, T is not onto.

(d) All multilinear functions $\delta\colon M_{n\times n}(\mathbf{R}) \to \mathbf{R}$ are equal to the determinant function $\det\colon M_{n\times n}(\mathbf{R}) \to \mathbf{R}$

FALSE: There are a few ways to see that this is false. Consider δ such that $\delta(A) := 2\det(A)$, where $A \in M_{n\times n}(\mathbf{R})$. Since det is multilinear, so is δ. However, δ is not equal to the determinant, since $\delta(I_n) = 2 \neq 1 = \det(I_n)$.

2. Question 2

Let $T\colon P_3(\mathbf{R}) \to \mathbf{R}^4$ be defined by $T(f) = (f(0), f(1), f'(0), f'(1))$. Let $\beta = (1, x, x^2, x^3)$ and let γ be the standard basis of $\mathbf{R}^4$.

(i) Calculate $[T]_\beta^\gamma$

Solution. Note that $T(1) = (1,1,0,0)$, $T(x) = (0,1,1,1)$, $T(x^2) = (0,1,0,2)$ and $T(x^3) = (0,1,0,3)$. So,

$$[T]_\beta^\gamma = ([T(1)]^\gamma, [T(x)]^\gamma, [T(x^2)]^\gamma, [T(x^3)]^\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

(ii) Is T an isomorphism? If yes, prove it. If not, explain why not.

Solution. We do some row reductions on $[T]_\beta^\gamma$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We claim that T is an isomorphism. To see this, note that there exist four elementary row operations E_1, E_2, E_3, E_4 such that $E_1E_2E_3E_4[T]_\beta^\gamma$ is in row-echelon form. Also, from a Lemma from the notes (Lemma 3.7, in the third set of notes), $E_1E_2E_3E_4[T]_\beta^\gamma$ has rank equal to its number of nonzero rows, which in this case is 4. From a Lemma from the notes (Lemma 3.10, in the third set of notes), since E_1, E_2, E_3, E_4 are invertible, we conclude that $[T]_\beta^\gamma$ also has rank 4. A 4×4 matrix of rank 4 is invertible, i.e. $[T]_\beta^\gamma$ is invertible. To see this, recall that $[T]_\beta^\gamma$ has only the zero vector in its null space by the rank-nullity theorem; also by the rank-nullity theorem, $[T]_\beta^\gamma$ is onto. So, $[T]_\beta^\gamma$ is both one-to-one and onto, so it is invertible by a Lemma in the notes (Lemma 6.6 from the second set of notes.) Finally, since $[T]_\beta^\gamma$ is invertible, we conclude that T is invertible, by a theorem from the notes (Theorem 6.11 in the second set of notes).

3. Question 3

(i) Let V, W be finite-dimensional vector spaces of the same dimension n. Prove that a one-to-one linear transformation $T: V \to W$ must be an isomorphism.

Solution. Since T is one-to-one, T only has the zero vector in its null space. By the rank-nullity Theorem, T has rank n. By a Lemma in the notes (Lemma 3.1 from the third set of notes), T is therefore invertible.

(ii) Let V be a finite-dimensional vector space and let $T: V \to V$ be a linear transformation. Suppose that, for some ordered basis β of V, we have $\det([T]_\beta^\beta) = 0$.

(a) Prove that $\det([T]_\gamma^\gamma) = 0$ for an arbitrary ordered basis γ of V.

Solution. Define $Q := [I_V]_\beta^\gamma$. Recall that (from Lemma 7.3 in the second set of notes), Q is an invertible matrix, and

$$Q[T]_\beta^\beta Q^{-1} = [I_V]_\beta^\gamma [T]_\beta^\beta ([I_V]_\beta^\gamma)^{-1} = [I_V]_\beta^\gamma [T]_\beta^\beta [I_V]_\gamma^\beta = [T]_\gamma^\gamma$$

So, taking the determinant of both sides and using the multiplicative property of the determinant,

$$\det([T]_\gamma^\gamma) = \det(Q[T]_\beta^\beta Q^{-1}) = \det(Q)\det([T]_\beta^\beta)\det(Q^{-1}).$$

The right side is zero since $\det([T]_\beta^\beta) = 0$. Therefore, $\det([T]_\gamma^\gamma) = 0$, as desired.

(b) Prove that $\text{nullity}(T) > 0$.

Solution. Since $\det([T]_\beta^\beta) = 0$, we know that $[T]_\beta^\beta$ is not invertible (by Theorem 4.5 in the fourth set of notes). Since $[T]_\beta^\beta$ is not invertible, we know that T is not invertible (by Theorem 6.11 in the second set of notes). Since T is not invertible, T is not one-to-one. (If T were one-to-one, then T would also by onto, by the rank-nullity theorem, so T would be invertible, a contradiction.) Since T is not one-to-one, there exists a nonzero vector v such that $Tv = 0$. That is, the nullity of T is positive.

4. Question 4

Let $V \subseteq P_2(\mathbf{R})$ be the subspace of all $f \in P_2(\mathbf{R})$ such that $f(1) = 0$. Let $\beta = (v_1, v_2) = (x - 1, x^2 - x)$ be an ordered basis of V.

(i) Let $a, b \in \mathbf{R}$ with $ab \neq -2$. Let $\gamma = (-v_1 + av_2, bv_1 + 2v_2)$ be another ordered basis of V. Calculate the change of coordinate matrix from β to γ.

Solution. The change of coordinate matrix Q is defined as $Q = [I_V]_\beta^\gamma$. So, we have

$$Q = [I_V]_\beta^\gamma = ([v_1]^\gamma, [v_2]^\gamma).$$

Using $ab \neq -2$ so that we do not divide by zero, we have

$$v_1 = (-1 - ab/2)^{-1}(-v_1 + av_2) + (-1 - ab/2)^{-1}(-a/2)(bv_1 + 2v_2)$$

$$v_2 = (ab + 2)^{-1}b(-v_1 + av_2) + (ab + 2)^{-1}(bv_1 + 2v_2).$$

Therefore,

$$Q = \begin{pmatrix} (-1 - ab/2)^{-1} & (ab + 2)^{-1}b \\ (-1 - ab/2)^{-1}(-a/2) & (ab + 2)^{-1} \end{pmatrix}.$$

(ii) Let $a = -2$, $b = -1$ in part (i). Suppose $T \colon V \to V$ satisfies $[T]_\gamma^\gamma = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find $[T]_\beta^\beta$.

Solution. Recall that (from Lemma 7.3 in the second set of notes),

$$[T]_\beta^\beta = [I_V]_\gamma^\beta [T]_\gamma^\gamma [I_V]_\beta^\gamma = Q^{-1}[T]_\gamma^\gamma Q.$$

So, if $a = -2$ and $b = -1$, we have

$$[T]_\beta^\beta = Q^{-1}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} Q = \begin{pmatrix} -1/2 & -1/4 \\ -1/2 & 1/4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1/2 & -1/4 \\ -1/2 & 1/4 \end{pmatrix}.$$

5. Question 5

(a) State the prove the dimension theorem for a linear transformation $T \colon V \to W$. (I don't think I would ask this question on an exam, myself.)

Solution. See Theorem 3.9 in the second set of notes.

(b) (This question deals with dual bases, which is not formally a part of our class.)

6. Question 6

Let V, W, U be three finite dimensional vector spaces, and let α, β, γ be ordered bases for V, W and U respectively. Prove that, for all $S\colon W \to U$ linear and for all $T\colon V \to W$ linear, we have

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta.$$

(I don't think I would ask this question on an exam, myself.)

Solution. See Theorem 5.14 in the second set of notes.

7. Extra Credit

Let D be an $n \times n$ diagonal matrix whose (i,i) entry is equal to i, for all $1 \leq i \leq n$. Define a linear transformation $T\colon M_{n\times n}(\mathbf{R}) \to M_{n\times n}(\mathbf{R})$ by $T(A) = DA - AD$. Find a basis of $N(T)$ and determine the rank of T.

Solution. Let e_i be a standard basis vector of $\mathbf{R}^n$ for each $1 \leq i \leq n$. Suppose $T(A) = 0$. That is, $DA = AD$. Since $De_i = ie_i$, we have $iAe_i = ADe_i = DAe_i$. That is, Ae_i is an eigenvector of D with eigenvalue i. Since the eigenspace of D with eigenvalue i is one-dimensional and it is the span of e_i, we know that Ae_i is a multiple of e_i. That is, there exists $\lambda_i \in \mathbf{R}$ such that e_i is an eigenvector of A with eigenvalue λ_i. Since this is true for all $1 \leq i \leq n$, we conclude that $(e_1, \ldots, e_n)$ is a basis of $\mathbf{R}^n$ consisting of eigenvectors of A. From Lemma 3.15 in the fourth set of notes, we conclude that A itself is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. That is, $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Therefore, if $T(A) = 0$, then A is contained in the subspace of all diagonal $n \times n$ matrices. Note that this space is n-dimensional. That is, $\dim(N(T)) \leq n$. We will in fact show that $\mathrm{nullity}(T) = n$. To see this, consider the $n \times n$ diagonal matrix D_i whose i^{th} entry is 1, with all other entries 0. Then the set of matrices $(D_1, \ldots, D_n)$ is a linearly independent set. Moreover, we just verified that $D_i \in N(T)$ for all $1 \leq i \leq n$. (We can also observe directly that $AD_i = D_iA$, so $T(D_i) = 0$ for all $1 \leq i \leq n$.) Therefore, $\mathrm{nullity}(T) \geq n$. In conclusion, $\mathrm{nullity}(T) = n$, so that $(D_1, \ldots, D_n)$ is a basis for $N(T)$, by Corollary 6.14(e) in the first set of notes.

Finally, by the rank-nullity theorem, we have $\mathrm{rank}(T) = \dim(M_{n\times n}(\mathbf{R})) - \mathrm{nullity}(T) = n^2 - n = n(n-1)$.