115A Midterm 1 Solutions[1]

1. Question 1 (True/False)

(a) A vector space may have more than one zero vector.

False. Suppose 0 and $0'$ are both zero vectors. Since 0 is a zero vector, we have $0' = 0 + 0'$. Since $0'$ is a zero vector, we have $0 = 0 + 0'$. In conclusion $0 = 0'$, so the zero vector is unique.

(b) Let V be a vector space over a field $\mathbf{F}$ and let W be a subset of V . If W is a vector space over the field $\mathbf{F}$, then W is a subspace of V.

True. This was Proposition 5.5 from the notes (Subspace Equivalence).

(c) Let V be a vector space over a field $\mathbf{F}$. Then the intersection of any two subsets of V is a subspace of V.

False. Let W denote the vector $(1, 0)$ in $\mathbf{R}^2$. Then $W \subseteq \mathbf{R}^2$ and $W \cap W = W$, but W is not a subspace of V.

(d) Every vector space has a finite basis.

False. The vector space $P(\mathbf{R})$ of all polynomials of one real variable is an infinite dimensional vector space. That is, no finite basis exists for this space. One can prove this directly by contradiction. Suppose $p_1, \ldots, p_n$ is a basis for $P(\mathbf{R})$. Let N be the maximum degree of the polynomials $p_1, \ldots, p_n$. Then the monomial x^{N+1} is not in the span of $p_1, \ldots, p_n$. To see this, note that any linear combination of $p_1, \ldots, p_n$ satisfies $(d/dx)^N \sum_{i=1}^n \alpha_i p_i = 0$, whereas $(d/dx)^N x^{N+1} \neq 0$. So, we get a contradiction, so $P(\mathbf{R})$ does not have a finite basis.

2. Question 2 (True/False)

(a) There exists a vector space V over $\mathbf{R}$ such that V has dimension 100.

True. Consider for example $\mathbf{R}^{100}$.

(b) If a vector space V is n-dimensional, then every subset of V with more than n elements is a spanning set for V.

False. Let $V = \mathbf{R}^2$, and let $S = \{(1, 0), (2, 0), (3, 0)\}$. Then the span of S is equal to the span of the vector $(1, 0)$. That is, S has 3 elements, but the $(0, 1)$ is not in the span of S, so S is not a spanning set for V.

(c) Let V be a vector space over a field $\mathbf{F}$. Let U, W be subspaces of V. Then $U \cup W$ is a subspace of V.

False. Consider $V = \mathbf{R}^2$. Let U be the x-axis, and let W be the y-axis. Then U, W are subspaces of V, but $U \cup W$ is just the union of the x and y axes of $\mathbf{R}^2$. So, for example, $(1, 0) \in U$, $(0, 1) \in W$, but $(1, 0) + (0, 1) = (1, 1)$, which does not lie in U nor in W, so $(1, 1) \notin U \cup W$, so $U \cup W$ is not closed under vector addition, so $U \cup W$ is not a subspace of V.

(d) A set of three vectors in $\mathbf{R}^2$ can be linearly independent.

False. Since $\mathbf{R}^2$ has a basis of two elements $\{(1, 0), (0, 1)\}$, this assertion follows from Corollary 6.14(b) in the notes.

3. Question 3

Let $\beta = ((1,0),(0,1))$ be the standard ordered basis of $\mathbf{R}^2$. Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that

$$T(1,2) = (3,4)$$

$$T(1,1) = (1,1).$$

Compute $[T]_\beta^\beta$. (As usual, we consider $\mathbf{R}^2$ to be a vector space over $\mathbf{R}$.)

Solution. Since T is linear, we have $T(1,0) = T(-(1,2)+2(1,1)) = -T(1,2)+2T(1,1) = -(3,4)+2(1,1) = (-1,-2)$. That is, $T(1,0) = -1(1,0)-2(0,1)$. Also, since T is linear, we have $T(0,1) = T((1,2)-(1,1)) = T(1,2)-T(1,1) = (3,4)-(1,1) = (2,3) = 2(1,0)+3(0,1)$. So, combining everything,

$$[T]_\beta^\beta = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

4. Question 4

Let $T: V \to V$ be a linear transformation on a vector space V. Assume that $\dim(V) = 7$, and that $T(T(v)) = 0$ for all $v \in V$. Prove or disprove the following statement: It is possible that T has rank 4. (Hint: one of the following is contained in the other: $R(T)$, $N(T)$.)

Solution. It is not possible that T has rank 4. To see this, note that since $T(T(v)) = 0$ for all $v \in V$, we have $R(T) \subseteq N(T)$. (If $y \in R(T)$, then $y = T(x)$ for some $x \in V$, and then $T(y) = T(T(x)) = 0$ by assumption, so that $y \in N(T)$.) Since $R(T) \subseteq N(T)$, we know that $\dim(R(T)) \leq \dim(N(T))$. From the rank-nullity theorem, $\dim(R(T)) + \dim(N(T)) = \dim(V) = 7$. Since $\dim(R(T)) \leq \dim(N(T))$ we get $2\dim(R(T)) \leq 7$, so $\dim(R(T)) \leq 7/2$. Since $\dim(R(T))$ is an integer, we have $\dim(R(T)) \leq 3$. That is, it cannot be the case that $\dim(R(T)) = 4$.

5. Question 5

Let U, W be subspaces of a vector space V. Suppose $\dim(V) = 5$, $\dim(U) = 3$ and $\dim(W) = 3$. Show that there exists a nonzero vector $v \in V$ such that $v \in U$ and $v \in W$. (Hint: let (u_1, u_2, u_3) be a basis for U. Complete this to a basis $(u_1, \ldots, u_5)$ of V. (Why can we do this?) Let (w_1, w_2, w_3) be a basis for W. Then write w_1 as a linear combination of $u_1, \ldots, u_5$, and similarly for w_2 and w_3. Then eliminate the u_4, u_5 from the set of equations.)

Solution 1. We argue by contradiction. Assume $U \cap W = \{0\}$. From Exercise 12 on Homework 2, $\dim(U)+\dim(W) = \dim(U \cap W)+\dim(U+W) = \dim(U+W)$. Since $U+W$ is a subspace of V, $\dim(U+W) \leq \dim(V) = 5$ (using Theorem 7.1 from the first set of notes). That is, $3+3 \geq 5$, a contradiction. That is, $U \cap W$ has a nonzero vector.

Solution 2. First of all, we can complete the basis of U to a basis of V by Corollary 6.14(f) in the first set of notes. Now, since $u_1, \ldots, u_5$ is a basis for V, there exist scalars α_{ij} for all $1 \leq i \leq 5$, $1 \leq j \leq 3$ such that $w_j = \sum_{i=1}^{5} \alpha_{ij} u_i$. Since we have three equations in the five variables $u_1, \ldots, u_5$, we can add/subtract these equations to get one equation that does not involve u_4 or u_5. That is, after adding and subtracting these three equations, we get some expression of the form

$$w_1 + \beta_2 w_2 + \beta_3 w_3 = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.$$

where $\beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 \in \mathbf{F}$. The vector on the left is in W, and the vector on the right is in U. Therefore, if we define $v := w_1 + \beta_2 w_2 + \beta_3 w_3$, then $v \in U$ and $v \in W$. Lastly, $v \neq 0$ since if $v = 0$ then we would have a nontrivial linear combination of the elements (w_1, w_2, w_3) which is zero, while (w_1, w_2, w_3) is a basis for U, which is by definition linearly independent. We conclude therefore that $v \neq 0$, as desired.