# 4: EIGENVALUES, EIGENVECTORS, DIAGONALIZATION

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Contents



## 1. Review

**Lemma 1.1.** *Let $V$ be a finite-dimensional vector space over a field $\mathbf{F}$. Let $\beta,\beta'$ be two bases for $V$. Let $T\colon V\to V$ be a linear transformation. Define $Q:=[I_V]_\beta^{\beta'}$. Then $[T]_\beta^\beta$ and $[T]_{\beta'}^{\beta'}$ satisfy the following relation*

$$[T]_{\beta'}^{\beta'}=Q[T]_\beta^\beta Q^{-1}.$$

**Theorem 1.2.** *Let $A$ be an $n\times n$ matrix. Then $A$ is invertible if and only if* $\det(A)\neq 0$.

**Exercise 1.3.** Let $A$ be an $n\times n$ matrix with entries $A_{ij}$, $i,j\in\{1,\ldots,n\}$, and let $S_n$ denote the set of all permutations on $n$ elements. For $\sigma\in S_n$, let $\mathrm{sign}(\sigma):=(-1)^N$, where $\sigma$ can be written as a composition of $N$ transpositions. Then

$$\det(A)=\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\prod_{i=1}^n A_{i\sigma(i)}.$$

## 2. Diagonal Matrices

So far, we should have a reasonably good understanding of linear transformations, matrices, rank and invertibility. However, given a matrix, we don't yet have a good understanding of how to "simplify" this matrix. In mathematics and science, the general goal is to take some complicated and make it simpler. In the context of linear algebra, this paradigm becomes: try to find a particular basis such that a linear transformation has a diagonal matrix representation. (After all, diagonal matrices are among the simplest matrices.) We now attempt to realize this goal within our discussion of eigenvectors and diagonalization.

**Definition 2.1** (**Diagonal Matrix**)**.** An $n\times n$ matrix $A$ with entries $A_{ij}$, $i,j\in\{1,\ldots,n\}$ is said to be **diagonal** if $A_{ij}=0$ whenever $i\neq j$, $i,j\in\{1,\ldots,n\}$. If $A$ is diagonal, we denote the matrix $A$ by $\mathrm{diag}(A_{11},A_{22},\ldots,A_{nn})$.

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**Lemma 2.2.** *The rank of a diagonal matrix is equal to the number of its nonzero entries.*

## 3. Eigenvectors and Eigenvalues

**Definition 3.1** (**Eigenvector and Eigenvalue**)**.** Let $V$ be a vector space over a field $\mathbf{F}$. Let $T\colon V \to V$ be a linear transformation. An **eigenvector** of $T$ is a nonzero vector $v \in V$ such that, there exists $\lambda \in \mathbf{F}$ with $T(v) = \lambda v$. The scalar $\lambda$ is then referred to as the **eigenvalue** of the eigenvector $v$.

**Remark 3.2.** The word "eigen" is German for "self." The equation $T(v) = \lambda v$ is self-referential in $v$, which explains the etymology here.

**Example 3.3.** If $A$ is diagonal with $A = \text{diag}(A_{11}, \ldots, A_{nn})$, then $L_A$ has eigenvectors $(e_1, \ldots, e_n)$ with eigenvalues $(A_{11}, \ldots, A_{nn})$.

**Example 3.4.** If $T$ is the identity transformation, then every vector is an eigenvector with eigenvalue 1.

**Example 3.5.** If $T\colon V \to V$ has $v \in N(T)$ with $v \neq 0$, then $v$ is an eigenvector of $T$ with eigenvalue zero.

**Example 3.6.** Define $T\colon C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$ by $T(f) := -f''$. For any $y \in \mathbf{R}$, the function $f(x) := e^{ixy}$ satisfies $Tf(x) = f''(x) = y^2 f(x)$. So, for any $y \in \mathbf{R}$, $e^{ixy}$ is an eigenfunction of $T$ with eigenvalue $y^2$.

**Definition 3.7** (**Eigenspace**)**.** Let $V$ be a vector space over a field $\mathbf{F}$. Let $T\colon V \to V$ be a linear transformation. Let $\lambda \in \mathbf{F}$. The **eigenspace** of $\lambda$ is the set of all $v \in V$ (including zero) such that $T(v) = \lambda v$.

**Remark 3.8.** Given $\lambda \in \mathbf{F}$, the set of $v$ such that $T(v) = \lambda v$ is the same as $N(T - \lambda I_V)$. In particular, an eigenspace is a subspace of $V$. And $N(T - \lambda I_V)$ is nonzero if and only if $T - \lambda I_V$ is not one-to-one.

**Lemma 3.9** (**An Eigenvector Basis Diagonalizes** $T$)**.** *Let $V$ be an $n$-dimensional vector space over a field $\mathbf{F}$, and let $T\colon V \to V$ be a linear transformation. Suppose $V$ has an ordered basis $\beta := (v_1, \ldots, v_n)$. Then $v_i$ is an eigenvector of $T$ with eigenvalue $\lambda_i \in \mathbf{F}$, for all $i \in \{1, \ldots, n\}$, if and only if the matrix $[T]_\beta^\beta$ is diagonal with $[T]_\beta^\beta = \text{diag}(\lambda_1, \ldots, \lambda_n)$.*

*Proof.* We begin with the forward implication. Let $i \in \{1, \ldots, n\}$. Suppose $T(v_i) = \lambda_i v_i$, $[T(v_i)]^\beta$ is a column vector whose $i^{th}$ entry is $\lambda_i$, with all other entries zero. Since $[T]_\beta^\beta = ([T(v_1)]^\beta, \ldots, [T(v_n)]^\beta)$, we conclude that $[T]_\beta^\beta = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

Conversely, suppose $[T]_\beta^\beta = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Since $[T]_\beta^\beta = ([T(v_1)]^\beta, \ldots, [T(v_n)]^\beta)$, we conclude that $T(v_i) = \lambda_i v_i$ for all $i \in \{1, \ldots, n\}$, so that $v_i$ is an eigenvector of $T$ with eigenvalue $\lambda_i$, for all $i \in \{1, \ldots, n\}$. □

**Definition 3.10** (**Diagonalizable**)**.** A linear transformation $T\colon V \to V$ is said to be **diagonalizable** if there exists an ordered basis $\beta$ of $V$ such the matrix $[T]_\beta^\beta$ is diagonal.

**Remark 3.11.** From Lemma 3.9, $T$ is diagonalizable if and only if it has a basis consisting of eigenvectors of $T$.

**Example 3.12.** Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ denote reflection across the line $\ell$ which passes through the origin and $(1,2)$. Then $T(1,2) = (1,2)$, and $T(2,-1) = -(2,-1)$, so we have two eigenvectors of $T$ with eigenvalues $1$ and $-1$ respectively. The vectors $((1,2),(2,-1))$ are independent, so they form a basis of $\mathbf{R}^2$. From Lemma 3.9, $T$ is diagonalizable. For $\beta := ((1,2),(2,-1))$, we have

$$[T]_\beta^\beta = \mathrm{diag}(1,-1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $[T^2]_\beta^\beta = I_2 = [I_{\mathbf{R}^2}]_\beta^\beta$, so $T^2 = I_{\mathbf{R}^2}$. The point of this example is that, once we can diagonalize $T$, taking powers of $T$ becomes very easy.

**Definition 3.13 (Diagonalizable Matrix).** An $n \times n$ matrix $A$ is **diagonalizable** if the corresponding linear transformation $L_A$ is diagonalizable.

**Lemma 3.14.** *A matrix $A$ is diagonalizable if and only if there exists an invertible matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^{-1}$. That is, a matrix $A$ is diagonalizable if and only if it is similar to a diagonal matrix.*

*Proof.* Suppose $A$ is an $n \times n$ diagonalizable matrix. Let $\beta$ denote the standard basis of $\mathbf{F}^n$, so that $A = [L_A]_\beta^\beta$. Since $A$ is diagonalizable, there exists an ordered basis $\beta'$ such that $D := [L_A]_{\beta'}^{\beta'}$ is diagonal. From Lemma 1.1, there exists an invertible matrix $Q := [I_{\mathbf{F}^n}]_{\beta'}^{\beta}$ such that

$$A = [L_A]_\beta^\beta = Q[L_A]_{\beta'}^{\beta'}Q^{-1} = QDQ^{-1}.$$

We now prove the converse. Suppose $A = QDQ^{-1}$, where $Q$ is invertible and $D$ is diagonal. Let $\lambda_1, \ldots, \lambda_n$ such that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then $De_i = \lambda_i e_i$ for all $i \in \{1, \ldots, n\}$, so

$$A(Qe_i) = QDQ^{-1}Qe_i = QDe_i = \lambda_i Qe_i.$$

So, $Qe_i$ is an eigenvector of $A$, for each $i \in \{1, \ldots, n\}$. Since $Q$ is invertible and $(e_1, \ldots, e_n)$ is a basis of $\mathbf{F}^n$, we see that $(Qe_1, \ldots, Qe_n)$ is also a basis of $\mathbf{F}^n$. So, $\beta'' := (Qe_1, \ldots, Qe_n)$ is a basis of $\mathbf{F}^n$ consisting of eigenvectors of $A$, so $A$ is diagonalizable by Lemma 3.9, since $[L_A]_{\beta''}^{\beta''}$ is diagonal. □

**Lemma 3.15.** *Let $A$ be an $n \times n$ matrix. Suppose $\beta' = (v_1, \ldots, v_n)$ is an ordered basis of $\mathbf{F}^n$ such that $v_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$ for all $i \in \{1, \ldots, n\}$. Let $Q$ be the matrix with columns $v_1, \ldots, v_n$ (where we write each $v_i$ in the standard basis). Then*

$$A = Q\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)Q^{-1}.$$

*Proof.* Let $\beta$ be the standard basis of $\mathbf{F}^n$. Note that $[I_{\mathbf{F}^n}]_{\beta'}^{\beta} = Q$. So, by Lemma 1.1,

$$A = [L_A]_\beta^\beta = Q[L_A]_{\beta'}^{\beta'}Q^{-1}.$$

Since $L_A v_i = \lambda_i v_i$ for all $i \in \{1, \ldots, n\}$,

$$[L_A]_{\beta'}^{\beta'} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

The Lemma follows. □

## 4. Characteristic Polynomial

**Lemma 4.1.** *Let $A$ be an $n \times n$ matrix. Then $\lambda \in \mathbf{F}$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I_n) = 0$.*

*Proof.* Suppose $\lambda$ is an eigenvalue of $A$. Then there exists $v \in \mathbf{F}^n$ such that $Av = \lambda v$ and $v \neq 0$, so that $(A - \lambda I_n)v = 0$. So, $(A - \lambda I_n)$ is not invertible, and $\det(A - \lambda I_n) = 0$, from the contrapositive of Theorem 1.2. Conversely, if $\det(A - \lambda I_n) = 0$, then $A - \lambda I_n$ is not invertible, from the contrapositive of Theorem 1.2. In particular, $A - \lambda I_n$ is not one-to-one. So, there exists $v \in \mathbf{F}^n$ with $v \neq 0$ such that $(A - \lambda I_n)v = 0$, i.e. $Av = \lambda v$. □

**Definition 4.2 (Characteristic Polynomial).** Let $A$ be an $n \times n$ with entries in a field $\mathbf{F}$. Let $\lambda \in \mathbf{F}$, and define the **characteristic polynomial** $f(\lambda)$ of $A$, by

$$f(\lambda) := \det(A - \lambda I_n).$$

**Lemma 4.3.** *Let $A, B$ be similar matrices. Then $A, B$ have the same characteristic polynomial.*

*Proof.* Let $\lambda \in \mathbf{F}$. Since $A, B$ are similar, there exists an invertible matrix $Q$ such that $A = QBQ^{-1}$. So, using the multiplicative property of the determinant,

$$\begin{aligned}\det(A - \lambda I) &= \det(QBQ^{-1} - \lambda I) = \det(Q(B - \lambda I)Q^{-1}) \\ &= \det(Q)\det(B - \lambda I)\det(Q^{-1}) = \det(Q)\det(Q)^{-1}\det(B - \lambda I) \\ &= \det(B - \lambda I).\end{aligned}$$

□

**Lemma 4.4.** *Let $A$ be an $n \times n$ matrix all of whose entries lie in $P_1(\mathbf{F})$. Then $\det(A) \in P_n(\mathbf{F})$.*

*Proof.* From Exercise 1.3 from the homework, $\det(A)$ is a sum of polynomials of degree at most $n$. That is, $\det(A)$ itself is in $P_n(\mathbf{R})$. □

**Remark 4.5.** From this Lemma, we see that the characteristic polynomial of $A$ is a polynomial of degree at most $n$.

**Lemma 4.6.** *Let $A$ be an $n \times n$ matrix with entries $A_{ij}$, $i, j \in \{1, \ldots, n\}$. Then there exists $g \in P_{n-2}(\mathbf{F})$ such that*

$$f(\lambda) = \det(A - \lambda I) = (A_{11} - \lambda) \cdots (A_{nn} - \lambda) + g(\lambda)$$

*Proof.* Let $B := A - \lambda I$. From Exercise 1.3 from the homework,

$$\det(A - \lambda I) = \prod_{i=1}^{n}(A_{ii} - \lambda) + \sum_{\sigma \in S_n \colon \sigma \neq I_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} B_{i\sigma(i)}.$$

Note that each term in the sum on the right has a number of $\lambda$ terms equal to the number of $i \in \{1, \ldots, n\}$ such that $i = \sigma(i)$. So, if $\sigma \in S_n$ and $\sigma \neq I_n$, it suffices to show that there exist at least two integers $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $\sigma(i) \neq i$ and $\sigma(j) \neq j$. We prove this assertion by contradiction. Suppose there exists $\sigma \in S_n$, $\sigma \neq I_n$ with exactly one $i \in \{1, \ldots, n\}$ with $\sigma(i) \neq i$. Then $\sigma(k) = k$ for all $k \in \{1, \ldots, n\} \smallsetminus \{i\}$. Since $\sigma$ is a permutation, $\sigma$ is onto, so there exists $i' \in \{1, \ldots, n\}$ such that $\sigma(i') = i$. Since $\sigma(k) = k$

for all $k \in \{1, \ldots, n\} \setminus \{i\}$, we must therefore have $i = i'$, so that $\sigma(i) = i$, a contradiction. We conclude that since $\sigma \neq I_n$, there exist at least two $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $\sigma(i) \neq i$ and $\sigma(j) \neq j$, as desired. $\square$

**Definition 4.7 (Trace).** Let $A$ be an $n \times n$ matrix with entries $A_{ij}$, $i, j \in \{1, \ldots, n\}$. Then the **trace** of $A$, denoted by $\mathrm{Tr}(A)$, is defined as

$$\mathrm{Tr}(A) := \sum_{i=1}^{n} A_{ii}.$$

**Theorem 4.8.** *Let $A$ be an $n \times n$ matrix. There exist scalars $a_1, \ldots, a_{n-2} \in \mathbf{F}$ such that the characteristic polynomial $f(\lambda)$ of $A$ satisfies*

$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}\mathrm{Tr}(A)\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + \det(A).$$

*Proof.* From Lemma 4.6, there exists $g \in P_{n-2}(\mathbf{F})$ such that

$$f(\lambda) = (A_{11} - \lambda) \cdots (A_{nn} - \lambda) + g(\lambda).$$

Multiplying out the product terms, we therefore get the two highest order terms of $f$. That is, there exists $G \in P_{n-2}(\mathbf{F})$ such that

$$f(\lambda) = (-\lambda)^n + \mathrm{Tr}(A)(-\lambda)^{n-1} + G(\lambda).$$

Finally, to get the zeroth order term of the polynomial $f$, note that by definition of the characteristic polynomial, $f(0) = \det(A)$. $\square$

**Example 4.9.** Let $a, b, c, d \in \mathbf{R}$. Then the characteristic polynomial of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$(a - \lambda)(d - \lambda) - bc = \lambda^2 - \lambda(a + d) + (ad - bc) = \lambda^2 - \lambda\mathrm{Tr}(A) + \det(A).$$

**Example 4.10.** The characteristic polynomial of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$.

**Example 4.11.** Let $i := \sqrt{-1}$. The characteristic polynomial of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is $\lambda^2 + 1 = (\lambda + i)(\lambda - i)$. However, we cannot factor $\lambda^2 + 1$ using only real numbers. So, as we will see below, we can diagonalize this matrix over the complex numbers, but not over the real numbers.

**Theorem 4.12 (The Fundamental Theorem of Algebra).** *Let $f(\lambda)$ be a real polynomial of degree $n$. Then there exist $\lambda_0, \lambda_1, \ldots \lambda_n \in \mathbf{C}$ such that*

$$f(\lambda) = \lambda_0 \prod_{i=1}^{n} (\lambda - \lambda_i).$$

**Remark 4.13.** This theorem is one of the reasons that complex numbers are useful. If we have complex numbers, then any real matrix has a characteristic polynomial that can be factored into complex roots. Without complex numbers, we could not do this.

## 5. Diagonalizability

Recall that $n$ linearly independent vectors in $\mathbf{F}^n$ form a basis of $\mathbf{F}^n$. So, from Lemma 3.9 or the proof of Lemma 3.14, we have

**Lemma 5.1.** *Let $A$ be an $n \times n$ matrix with elements in $\mathbf{F}$. Then $A$ is diagonalizable (over $\mathbf{F}$) if and only if there exists a set $(v_1, \ldots, v_n)$ of linearly independent vectors in $\mathbf{F}^n$ such that $v_i$ is an eigenvector of $A$ for all $i \in \{1, \ldots, n\}$.*

We now examine some ways of finding a set of linearly independent eigenvectors of $A$, since this will allow us to diagonalize $A$.

**Proposition 5.2.** *Let $A$ be an $n \times n$ matrix. Let $v_1, \ldots, v_k$ be eigenvectors of $A$ with eigenvalues $\lambda_1, \ldots, \lambda_k$, respectively. If $\lambda_1, \ldots, \lambda_k$ are all distinct, then the vectors $v_1, \ldots, v_k$ are linearly independent.*

*Proof.* We argue by contradiction. Assume there exist $\alpha_1, \ldots, \alpha_k \in \mathbf{F}$ not all zero such that

$$\sum_{i=1}^{k} \alpha_i v_i = 0.$$

Without loss of generality, $\alpha_1 \neq 0$. Applying $(A - \lambda_k I)$ to both sides,

$$0 = \sum_{i=1}^{k-1} \alpha_i (A - \lambda_k I) v_i = \sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) v_i.$$

We now apply $(A - \lambda_{k-1} I)$ to both sides, and so on. Continuing in this way, we eventually get the equality

$$0 = \alpha_1 (\lambda_1 - \lambda_k)(\lambda_1 - \lambda_{k-1}) \cdots (\lambda_1 - \lambda_2) v_1.$$

Since $\lambda_1, \ldots, \lambda_k$ are all distinct, and $\alpha_1 \neq 0$, and since $v_1 \neq 0$ (since it is an eigenvector), we have arrived at a contradiction. We conclude that $v_1, \ldots, v_k$ are linearly independent. $\square$

**Corollary 5.3.** *Let $A$ be an $n \times n$ matrix with elements in $\mathbf{F}$. Suppose the characteristic polynomial $f(\lambda)$ of $A$ can be written as $f(\lambda) = \prod_{i=1}^{n} (\lambda_i - \lambda)$, where $\lambda_i \in \mathbf{F}$ are distinct, for all $i \in \{1, \ldots, n\}$. Then $A$ is diagonalizable.*

*Proof.* For all $i \in \{1, \ldots, n\}$, let $v_i \in \mathbf{F}^n$ be the eigenvector corresponding to the eigenvalue $\lambda_i$. Setting $k = n$ in Proposition 5.2 shows that $v_1, \ldots, v_n$ are linearly independent. Lemma 5.1 therefore completes the proof. $\square$

**Example 5.4.** Consider

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}.$$

The characteristic polynomial is then

$$f(\lambda) = (1 - \lambda)(4 - \lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

So, $f(\lambda)$ has two distinct real roots, and we can diagonalize $A$ over $\mathbf{R}$. Observe that $v_1 = (2,-1)$ is an eigenvector with eigenvalue 2 and $v_2 = (1,-1)$ is an eigenvector with eigenvalue 3. So, if we think of the eigenvectors as column vectors, and use them to define $Q$,

$$Q := \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix},$$

we then have the desired diagonalization

$$\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} = Q \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} Q^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

**Exercise 5.5.** Using the matrix from Example 4.11, find its diagonalization over $\mathbf{C}$.

In summary, if $A$ is an $n \times n$ matrix with elements in $\mathbf{F}$, and if we can write the characteristic polynomial of $A$ as a product of $n$ distinct roots in $\mathbf{F}$, then $A$ is diagonalizable over $\mathbf{F}$. On the other hand, if we cannot write the characteristic polynomial as a product of $n$ roots in $\mathbf{F}$, then $A$ is not diagonalizable over $\mathbf{F}$. (Combining Lemmas 3.14 and 4.3 shows that, if $A$ is diagonalizable, then it has the same characteristic polynomial as a diagonal matrix. That is, the characteristic polynomial of $A$ is the product of $n$ roots.) (Recalling Example 4.11, the real matrix with characteristic polynomial $\lambda^2 + 1$ can be diagonalized over $\mathbf{C}$ but not over $\mathbf{R}$.)

The only remaining case to consider is when the characteristic polynomial of $A$ can be written as a product of $n$ non-distinct roots of $\mathbf{F}$. Unfortunately, this case is more complicated. It can be dealt with, but we don't have to time to cover the entire topic. The two relevant concepts here would be the Jordan normal form and the minimal polynomial.

To see the difficulty, note that the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is diagonal, so it is diagonalizable. Also, the standard basis of $\mathbf{R}^2$ are eigenvectors, and the characteristic polynomial is $(2-\lambda)^2$.

On the other hand, consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

This matrix also has characteristic polynomial $(2-\lambda)^2$, but it is not diagonalizable. To see this, we will observe that the eigenvectors of $A$ do not form a basis of $\mathbf{R}^2$. Since 2 is the only eigenvalue, all of the eigenvectors are in the null space of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

However, this matrix has only a one-dimensional null space, which is spanned by the column vector $(1,0)$. Since the eigenvectors of $A$ do not form a basis of $\mathbf{R}^2$, $A$ is not diagonalizable, by Remark 3.11 (or Lemma 3.9).

## 6. Appendix: Notation

Let $A, B$ be sets in a space $X$. Let $m, n$ be a nonnegative integers. Let $\mathbf{F}$ be a field.

$$\begin{aligned}
\mathbf{Z} &:= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the integers} \\
\mathbf{N} &:= \{0, 1, 2, 3, 4, 5, \ldots\}, \text{ the natural numbers} \\
\mathbf{Q} &:= \{m/n \colon m, n \in \mathbf{Z}, n \neq 0\}, \text{ the rationals} \\
\mathbf{R} &\text{ denotes the set of real numbers} \\
\mathbf{C} &:= \{x + y\sqrt{-1} : x, y \in \mathbf{R}\}, \text{ the complex numbers} \\
\emptyset &\text{ denotes the empty set, the set consisting of zero elements} \\
\in &\text{ means "is an element of." For example, } 2 \in \mathbf{Z} \text{ is read as "2 is an element of } \mathbf{Z}\text{."} \\
\forall &\text{ means "for all"} \\
\exists &\text{ means "there exists"} \\
\mathbf{F}^n &:= \{(x_1, \ldots, x_n) \colon x_i \in \mathbf{F}, \forall\, i \in \{1, \ldots, n\}\} \\
A \subseteq B &\text{ means } \forall\, a \in A, \text{ we have } a \in B, \text{ so } A \text{ is contained in } B \\
A \smallsetminus B &:= \{x \in A \colon x \notin B\} \\
A^c &:= X \smallsetminus A, \text{ the complement of } A \\
A \cap B &\text{ denotes the intersection of } A \text{ and } B \\
A \cup B &\text{ denotes the union of } A \text{ and } B \\
C(\mathbf{R}) &\text{ denotes the set of all continuous functions from } \mathbf{R} \text{ to } \mathbf{R} \\
P_n(\mathbf{R}) &\text{ denotes the set of all real polynomials in one real variable of degree at most } n \\
P(\mathbf{R}) &\text{ denotes the set of all real polynomials in one real variable} \\
M_{m \times n}(\mathbf{F}) &\text{ denotes the vector space of } m \times n \text{ matrices over the field } \mathbf{F} \\
I_n &\text{ denotes the } n \times n \text{ identity matrix} \\
\det &\text{ denotes the determinant function} \\
S_n &\text{ denotes the set of permutations on } \{1, \ldots, n\} \\
\mathrm{sign}(\sigma) &:= (-1)^N \text{ where } \sigma \in S_n \text{ can be written as the composition of } N \text{ transpositions} \\
\mathrm{Tr} &\text{ denotes the trace function}
\end{aligned}$$

### 6.1. Set Theory.

**6.1. Set Theory.** Let $V, W$ be sets, and let $f \colon V \to W$ be a function. Let $X \subseteq V$, $Y \subseteq W$.

$$f(X) := \{f(v) \colon v \in V\}.$$

$$f^{-1}(Y) := \{v \in V \colon f(v) \in Y\}.$$

The function $f \colon V \to W$ is said to be **injective** (or **one-to-one**) if: for every $v, v' \in V$, if $f(v) = f(v')$, then $v = v'$.

The function $f \colon V \to W$ is said to be **surjective** (or **onto**) if: for every $w \in W$, there exists $v \in V$ such that $f(v) = w$.

The function $f\colon V \to W$ is said to be **bijective** (or a **one-to-one correspondence**) if: for every $w \in W$, there exists exactly one $v \in V$ such that $f(v) = w$. A function $f\colon V \to W$ is bijective if and only if it is both injective and surjective.

Two sets $X, Y$ are said to have the same **cardinality** if there exists a bijection from $V$ onto $W$.

The **identity map** $I\colon X \to X$ is defined by $I(x) = x$ for all $x \in X$. To emphasize that the domain and range are both $X$, we sometimes write $I_X$ for the identity map on $X$.

Let $V, W$ be vector spaces over a field $\mathbf{F}$. Then $\mathcal{L}(V, W)$ denotes the set of linear transformations from $V$ to $W$, and $\mathcal{L}(V)$ denotes the set of linear transformations from $V$ to $V$.

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