

**MTHED-UE-1049: Mathematical Proof and Proving (MPP)**  
**MATH-UA-125: Introduction to Mathematical Proofs**

**Homework No. 8**

This homework should be submitted just before the beginning of class, on April 9<sup>th</sup>, 2012.

You should bring to class a copy of the homework that you submit, or at least notes that can remind you of what you did, in order to participate in class discussions.

Note that sections \*2.10, \*2.11, and problem \*3 (including \*3.1 and \*3.2) are bonus questions. You do not have to answer them. If you do, you will get bonus points for correct answers.

1. In class we did some geometry proofs.

For the proofs you need to do in this part you may rely on the following 'facts':

- (i) If two sides of a triangle are equal, then the measures of the angles opposite these sides are equal (in other words, in an isosceles triangle the base angles are congruent/of equal measure).
- (ii) If two angles of a triangle are equal, then the measures of the sides opposite these angles are equal (this is the converse of the above).
- (iii) An exterior angle of a triangle is larger than any interior angle that is not adjacent to it.

In each proof, you must indicate exactly where you used these facts.

- 1.1 Write a full proof that: **If two sides of a triangle are unequal, then the measure of the angle opposite the larger side is larger than the measure of the angle opposite the smaller side.**
- 1.2 Write a full proof of the converse: **If two angles of a triangle are unequal, then the measure of the side opposite the larger angle is larger than the measure of the side opposite the smaller angle.**
- 1.3 Although we know that the straight line is the shortest distance between two points, let's see how we can use what you proved in 1.2 to show that the sum of two sides of a triangle is always larger than the third side.

A hint: Result 1.2 gives us a tool to compare two segments of a triangle. Take any triangle with side lengths  $a, b, c$ . Suppose we could construct another triangle with one side  $K$  of length  $a$ , and another of side  $L$  of length  $b+c$ , and such that the angle opposite  $L$  is larger than the angle opposite  $K$ . If so, then 1.2 would tell us that  $b+c > a$ . To construct such a triangle, take any given triangle with side lengths  $a, b, c$  and extend one side by adding a segment that has length equal to one of the other two sides

Write a full proof that:

**The sum of two sides of a triangle is always larger than the third side.**

Does your proof hold for any triangle? Does it hold for any two sides you choose? Explain.

2. Here are some questions regarding irrationality of numbers. In some cases, the proof is simpler if you use the results that were already established (by proving or disproving) in previous work or in preceding cases (sub-sections). Try to be as efficient as possible, and to not do a tedious proof when there is an easier/shorter way to go about it. You can also use results from previous homework.

2.1 In the class activity you were asked to find the flaw in a 'proof' to the false claim that  $\sqrt{4}$  is an irrational number. Clearly, this cannot be a valid proof, because we know for sure that  $\sqrt{4}$  is a rational number (since  $\sqrt{4} = 2$ ), thus it cannot be irrational. The question was: Where exactly did the flawed proof go wrong?

Please review the proof we did in class, that  $\sqrt{2}$  is an irrational number, and the proof you did as homework – that  $\sqrt{3}$  is an irrational number. Explain where exactly the proof for  $\sqrt{2}$  (or  $\sqrt{3}$ ) does not hold for  $\sqrt{4}$ .

2.2 Is  $\sqrt{6}$  an irrational number? Why or why not? Prove your claim.

2.3 Is the product of two irrational numbers always an irrational number? Why or why not? Prove your claim.

2.4 Is the number  $(\sqrt{2} + \sqrt{3})$  an irrational number? Why or why not? Prove your claim.

2.5 Is the sum of two irrational numbers always an irrational number? Why or why not? Prove your claim.

2.6 Is the number  $(\sqrt{2} + 8)$  an irrational number? Why or why not? Prove your claim.

2.7 Is the sum of an irrational number and a rational number always an irrational number? Why or why not? Prove your claim.

2.8 Is  $\frac{5}{9} \cdot \sqrt{2}$  an irrational number? Why or why not? Prove your claim.

2.9 Is the product of an irrational number and a rational number always an irrational number? Why or why not? Prove your claim.

\*2.10 What can you say about the set of numbers of the form:  $a + b\sqrt{2}$ , where  $a, b \in \mathbb{Q}$  (i.e.,  $a, b$  are Rational numbers),  $b \neq 0$ ? Are all or some of them irrational numbers? Why or why not? Prove your claim.

\*2.11 What can you say about the set of numbers of the form:  $a + b\sqrt{2}$ , where  $a, b \in \mathbb{Q}$  (i.e.,  $a, b$  are Rational numbers), and at least one of  $a, b$  is non-zero? Are all or some of them irrational numbers? Why or why not? Prove your claim.

\*3. In class we defined a Group  $(G, *)$ , where  $G$  is a set and  $*$  is an operation that satisfies the following four conditions:

- (i) If  $g_1, g_2 \in G$ , then  $g_1 * g_2 \in G$  (this means that  $*$  is a binary operation on  $G$ );
- (ii) If  $g_1, g_2, g_3 \in G$ , then  $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$  (this means that  $*$  is associative);
- (iii)  $\exists e \in G$ , for which  $e * g = g * e = g, \forall g \in G$  (this means that there is an element  $e$  in  $G$  that is an identity);
- (iv)  $\forall g \in G, \exists g' \in G$ , for which  $g * g' = g' * g = e$  (this means that every element in  $G$  has an 'inverse');

\*3.1 Give an example of a group, and prove that it satisfies the four conditions of the definition.

\*3.2 Let us designate by  $G$  the set of numbers defined in 2.11, and examine as an operation on  $G$  the multiplication  $\times$  of real numbers. Is  $(G, \times)$  a group? Prove your claim.