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MTHED-UE-1049: Mathematical Proof and Proving (MPP)
MATH-UA-125: Introduction to Mathematical Proofs

Homework No. 6

This homework should be submitted just before the beginning of class, on March 26th, 2012. You should bring a copy of your homework to class, in order to participate in class discussion around your homework.
Please read carefully thefollowing instructions:

This homework is based on questions from the midterm exam and on your responses to them.

For three problems (1, 6, 7) we offer a skeleton of a proof, and you are required to add all the missing parts, including the Given, the RTP, and the justification for each step.

For two problems (2, 3) we offer a full proof. For these problems we provide a sample of responses that have flaws, inaccuracies, and/or redundancy. You need to point to all the flaws, inaccuracies and redundant (unnecessary) steps in the responses and explain why you regard them as such.

1. Let $n \in \mathbb{Z}$. Prove that if $5n-7$ is even then n is odd.

Given:

RTP:

A skeleton of a proof:

$$5n-7=2k$$

$$5n=2k+7$$

$$odd \times n = odd$$

$$n = odd$$

6. Let $x, y \in Z$.

(a) Prove that $(x^2 - y^2)$ is divisible by 4 if x and y are of the same parity (i.e., either both x and y are even or both x and y are odd).

Given:

RTP:

A skeleton of a proof:

$$x + y = 2n$$

$$x - y = 2k$$

$$x^2 - y^2 = 4nk$$

$$\frac{x^2 - y^2}{4} = nk$$

7. (*a*) Prove that for any two **positive** numbers x , y, their arithmetic mean is larger than or equal to their geometric mean, i.e.: $\sqrt{x \cdot y} \le \frac{x+y}{2}$

Given:

RTP:

A skeleton of a proof:

$$(x-y)^2 \ge 0$$

$$x^2 + y^2 - 2xy \ge 0$$

$$x^2 + y^2 + 2xy \ge 4xy$$

$$(x+y)^2 \ge 4xy$$

$$\frac{x+y}{2} \ge \sqrt{xy}$$

(*b*) When are these two means equal? That is, under what conditions does $\sqrt{x \cdot y} = \frac{x+y}{2}$ for **positive** numbers x and y? Prove your claim.

Given:

RTP:

A skeleton of a proof:

$$\frac{x+y}{2} = \sqrt{xy}$$

$$x + y = 2\sqrt{xy}$$

$$x^2 + y^2 + 2xy = 4xy$$

$$(x-y)^2 = 0$$

$$x = y$$

2. Let $x, y \in R$. Prove that $|x \cdot y| = |x| \cdot |y|$.

Given: $x, y \in R$

RTP: $|x \cdot y| = |x| \cdot |y|$

A Proof: *We use the following definition of the Absolute Value of a real number r:*

$$|r| = \begin{cases} r, \text{ if } r \geq 0 \\ -r, \text{ if } r \leq 0 \end{cases}$$

There are 3 *cases we need to check* :

Case 1: $x \geq 0, y \geq 0$; **Case 2** : $x \geq 0, y \leq 0$; **Case 3** : $x \leq 0, y \leq 0$

Case 1 :

$$\left.\begin{array}{l} x \geq 0 \Rightarrow |x| = x \\ y \geq 0 \Rightarrow |y| = y \end{array}\right\} \Rightarrow |x| \cdot |y| = x \cdot y$$

$$x \geq 0, y \geq 0 \Rightarrow x \cdot y \geq 0 \Rightarrow |x \cdot y| = x \cdot y$$

$$\Downarrow$$

$$|x \cdot y| = |x| \cdot |y|$$

Case 2 :

$$\left.\begin{array}{l} x \geq 0 \Rightarrow |x| = x \\ y \leq 0 \Rightarrow |y| = -y \end{array}\right\} \Rightarrow |x| \cdot |y| = x \cdot (-y) = -(x \cdot y)$$

$$x \geq 0, y \leq 0 \Rightarrow x \cdot y \leq 0 \Rightarrow |x \cdot y| = -(x \cdot y)$$

$$\Downarrow$$

$$|x \cdot y| = |x| \cdot |y|$$

Case 3 :

$$\left.\begin{array}{l} x \leq 0 \Rightarrow |x| = -x \\ y \leq 0 \Rightarrow |y| = -y \end{array}\right\} \Rightarrow |x| \cdot |y| = (-x) \cdot (-y) = x \cdot y$$

$$x \leq 0, y \leq 0 \Rightarrow x \cdot y \geq 0 \Rightarrow |x \cdot y| = x \cdot y$$

$$\Downarrow$$

$$|x \cdot y| = |x| \cdot |y|$$

Q.E.D.

SAMPLE RESPONSES FOR PROBLEM 2:

Response 2.1:

Case 1

Let $x = -a$ and $y = -b$

$$|-a \cdot -b| = |-a| \cdot |-b|$$
$$a \cdot b = a \cdot b$$

Case 2

Let $x = -a$ and $y = b$

$$|-a \cdot b| = |-a| \cdot |b|$$
$$a \cdot b = a \cdot b$$

Case 3

Let $x = a$ and $y = b$

$$|a \cdot b| = |a| \cdot |b|$$
$$a \cdot b = a \cdot b$$

Response 2.2:

RTP: $|x \cdot y| = |x| \cdot |y|$

Proof:

$\Rightarrow (xy)^2 = x^2 \cdot y^2$ $(\mathit{square\ both\ sides\ and\ get\ rid\ of\ the\ absolute\ value})$

$\Rightarrow \sqrt{(xy)^2} = \sqrt{x^2 \cdot y^2}$ $(\mathit{take\ square\ root\ to\ get\ rid\ of\ squared})$

$\Rightarrow xy = \sqrt{x^2} \cdot \sqrt{y^2}$

$\Rightarrow xy = x \cdot y$

✓

Response 2.3:

Given: $x, y \in R$

RTP: $|x \cdot y| = |x| \cdot |y|$

Proof: When you multiply x and y, depending on what they are equal to, you may get a positive or negative answer. The absolute value of $x \cdot y$ will ensure that the answer is positive.

Example:

$|5 \cdot 6| = 30$; $|5| \cdot |6| = 30$

$|-5 \cdot -6| = 30$; $|-5| \cdot |-6| = 30$

$|5 \cdot -6| = 30$; $|5| \cdot |-6| = 30$

$|-5 \cdot 6| = 30$; $|-5| \cdot |6| = 30$

When you put absolute value brackets around x and y separately, this makes both x and y positive factors, which must result in the same positive value that $|x \cdot y|$ gives you.

3. (a) Prove that $n^3 - 3n^2 - 9 \geq 0$ for $n \geq 6$, $n \in N$.

(b) Does this inequality hold for $n > 6$, $n \in N$? Why?

(c) Does this inequality hold for $n \geq 10$, $n \in N$? Why?

(d) Does this inequality hold for $n \geq 4$, $n \in N$? Why?

(e) Does this inequality hold for $n \geq 2$, $n \in N$? Why?

Part (a):

Given: $n \geq 6$, $n \in N$

RTP: $n^3 - 3n^2 - 9 \geq 0$

A Proof:

$n^3 = n \cdot n^2$ (*follows from the definition of a power of n*) $\Rightarrow$ $n^3 - 3n^2 - 9 = n \cdot n^2 - 3n^2 - 9 = (n-3) \cdot n^2 - 9$

$n \geq 6$ ($given$)

$\Downarrow$

$n - 3 \geq 6 - 3 = 3$, $n^2 \geq 36$

$\Downarrow$

$(n-3) \cdot n^2 - 9 \geq 3 \cdot 36 - 9 \geq 0$

$\Downarrow$

$n^3 - 3n^2 - 9 \geq 0$

Q.E.D.

Part (b):

Yes. $n > 6$, $n \in N$ is included in $n \geq 6$, $n \in N$, and we proved the inequality for $n \geq 6$, $n \in N$.

Part (c):

Yes. $n \geq 10$, $n \in N$ is included in $n \geq 6$, $n \in N$, and we proved the inequality for $n \geq 6$, $n \in N$.

Part (d):

Yes. For $n = 4$, $n^3 - 3n^2 - 9 = 7 \geq 0$ and for $n = 5$, $n^3 - 3n^2 - 9 = 41 \geq 0$.

We proved the inequality for $n \geq 6$, $n \in N$ and showed that it holds for $n = 4$ and for $n = 5$, thus it holds for $n \geq 4$, $n \in N$.

Part (e):

No. For $n = 3$ the inequality does not hold: $n^3 - 3n^2 - 9 = -9 < 0$.

SAMPLE RESPONSES FOR PROBLEM 3:

Response 3.1:

(a) Let $n = 6$.

$$6^3 - 3 \cdot 6^2 - 9 \geq 0$$
$$99 \geq 0$$

(c) Let $n = 10$.

$$10^3 - 3 \cdot 10^2 - 9 \geq 0$$
$$691 \geq 0$$

(d) Let $n = 4$.

$$4^3 - 3 \cdot 4^2 - 9 \geq 0$$
$$7 \geq 0$$

Response 3.2:

(a) Given: $n \geq 6$.

Prove: $n^3 - 3n^2 - 9 \geq 0$

$n^3 - 3n^2 \geq 9$ $(add\ 9\ to\ both\ sides)$

$n^2(n-3) \geq 9$ $(\ factor\ out\ n^2)$

$6^2(6-3) \geq 9$ $(substitute\ 6\ for\ n, because\ 6\ is\ the\ lowest\ possible\ number\ for\ n,$
$so\ if\ it's \geq, then\ any\ number\ above\ 6\ will\ be \geq also)$

$36 \cdot (3) \geq 9$

✓

(c) Yes, because the inequality is true when $n \geq 6$, so it will only get bigger, the larger n is.

<u>Response 3.3</u>:

(a) $n^3 - 3n^2 - 9 \geq 0$ for $n \geq 6$.

Proving by using the contrapositive of the statement:

The contrapositive is $n < 6$ for $n^3 - 3n^2 - 9 < 0$.

$n < 6$, $n \in N \Rightarrow n \cdot n^2 < 6n^2 \Rightarrow n^3 < 6n^2 \Rightarrow n^3 - 6n^2 < 0$

$\Rightarrow n^3 - 6n^2 + 3n^2 < 3n^2 \Rightarrow n^3 - 3n^2 < 3n^2 \Rightarrow n^3 - 3n^2 - 9 < 3n^2 - 9$

$\Rightarrow n^3 - 3n^2 - 9 < 3(n^2 - 1) \Rightarrow n^3 - 3n^2 - 9 < 3(n-1)(n+1)$

(d) For $n \geq 4$?

No, the inequality doesn't hold for $n \geq 4$ because 5 is not in the original statement.
However, values for $n \geq 4$ will still make the inequality greater than 0.