Department of Teaching and Learning, Steinhardt School of Culture, Education, & Human Development
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MTHED-UE-1049: Mathematical Proof and Proving (MPP)
MATH-UA-125: Introduction to Mathematical Proofs

Homework No. 3

This homework should be submitted just before the beginning of class, on Feruary 27th, 2012.
Please write in a black ink pen, so it is clear and easy to read! Use white paper!!
Write your name in Capital letters on the top of each page and number the pages.

1. In class we proved, without using a diagram, that for any two sets A and B: $(A \cap B)^c = A^c \cup B^c$.
   Use a similar method to prove that for any two sets A and B: $(A \cup B)^c = A^c \cap B^c$

2. Prove that for any x that is an integer, if ($x^2 - 1$) is not divisible by 3, then x is divisible by 3.
   Hint: It is easier to prove the contrapositive.

3. Prove that for any rational number n, if $n^5 - 6n^4 + 27 < 0$, then $n \le 10$.

4. The "divisibility-by-3 rule" says that:
   If the **sum** of the digits of an integer n is divisible by 3, then n is divisible by 3.
   (a) Prove the "divisibility-by-3 rule" for 4-digit integers.
   (b) What is the converse of the "divisibility-by-3 rule"?
   (c) Prove or disprove the converse of "the divisibility-by-3 rule" for 4-digit integers.

5. Find a mathematical implication that is **true**, for which its converse is also **true**:
   (a) Formulate the statement as "if …., then …".
   (b) Prove that it is true.
   (c) Formulate its converse.
   (d) Prove that the converse is true.

6. Find a mathematical implication that is **true**, for which its converse is **false**:
   (a) Formulate the statement as "if …., then …".
   (b) Prove that it is true.
   (c) Formulate its converse.
   (d) Prove that the converse is false.

7. Find another mathematical implication that is **true**, for which its converse is **false**:
   (a) Formulate the statement as "if …., then …".
   (b) Prove that it is true.
   (c) Formulate its converse.
   (d) Prove that the converse is false.

8. In class we discussed the following statement:

(*) "If an integer n is a perfect square (i.e., n is a square of some integer), then it has an odd number of divisors." Note that we include 1 and n as divisors of n.

You found several examples that satisfy the statement, that is, perfect squares that have an odd number of divisors.

You were not able to find an example that contradicts it.

Thus – you felt that the statement is probably true.

(a) Prove that the above statement (*) is indeed true.

(b) You were able to find examples of integers that have an even number of divisors that are not perfect squares (e.g., 10). Explain how this finding is related to the contrapositive of the statement (*).

(c) However, you were not able to find an example of an integer that has an odd number of divisors and is not a perfect square. What conjecture(s) can this observation lead to?

(d) Formulate the converse of the above statement (*).

(e) Do you think the converse of (*) is true? Explain why you think so.