

MTHED-UE-1049: Mathematical Proof and Proving (MPP)
MATH-UA-125: Introduction to Mathematical Proofs

Homework No. 11

This homework should be submitted just before the beginning of class, on April 30th, 2012.
 You should bring to class a copy of the homework that you submit, or at least notes that can remind you of what you did, in order to participate in class discussions.
 You must justify all answers and all steps in your proofs.

Part 1: An example of a full proof by mathematical induction (read carefully and make sure you follow):

Prove that $1+2+3+\dots+n = \frac{n \cdot (n+1)}{2}$, $\forall n \in \mathbb{N}$:

a. Base Step (verifying for $n = 1$):

We check the value of each side for $n = 1$:

Left Side (L.S.) = 1

$$\text{Right Side (R.S.)} = \frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1$$

\Rightarrow Both Sides are Equal (to 1) ✓

If we want, we can also check the value of each side for $n = 2$:

Left Side (L.S.) = $1 + 2 = 3$

$$\text{Right Side (R.S.)} = \frac{2 \cdot (2+1)}{2} = \frac{6}{2} = 3$$

\Rightarrow Both Sides are Equal (to 3) ✓

b. The Inductive Step (Mini-Proof):

Given: The equality is true for $n = k$, $k \in \mathbb{N}$, that is, $1+2+3+\dots+k = \frac{k \cdot (k+1)}{2}$

RTP: The equality is true for $n = k+1$, that is, $1+2+3+\dots+k+(k+1) = \frac{(k+1) \cdot (k+2)}{2}$

The Proof (of the Inductive Step):

Note: We start with the Left Side of the RTP, and by logical inference show that it is equal to the Right Side of the RTP.

$$\underbrace{1+2+3+\dots+k+(k+1)}_{\text{L.S. of RTP}} \stackrel{\text{Based on the Given}}{=} \frac{k \cdot (k+1)}{2} + (k+1) \stackrel{\text{Common Denominator}}{=} \frac{k \cdot (k+1) + 2 \cdot (k+1)}{2} \stackrel{\text{Algebraic Manipulations}}{=} \frac{(k+1) \cdot (k+2)}{2} \stackrel{\text{R.S. of RTP}}{=} \frac{(k+1) \cdot (k+2)}{2}$$

$\Rightarrow 1+2+3+\dots+k+(k+1) = \frac{(k+1) \cdot (k+2)}{2}$ Q.E.D. (the mini-proof)

From a and b, we conclude by Mathematical Induction that: $1+2+3+\dots+n = \frac{n \cdot (n+1)}{2}$, $\forall n \in \mathbb{N}$. Q.E.D.

Part 2: The following proof is not complete. Copy the proof and fill in the missing parts and justifications (where there are ???):

Prove that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$, $\forall n \in N$:

a. Base Step (verifying for $n = 1$):

We check the value of each side for $n = 1$:

Left Side (L.S.)=???

Right Side (R.S.)= ???

\Rightarrow Both Sides are Equal (to ???)

If you want, you can also check the value of each side for $n = 2$ too:

Left Side (L.S.)= $1^2 + 2^2 = 5$

Right Side (R.S.)= ???

\Rightarrow Both Sides are Equal (to ???)✓

b. The Inductive Step (Mini-Proof):

Given: The equality is true for $n = k$, $k \in N$, that is, $1^2 + 2^2 + 3^2 + \dots + k^2 = ???$

RTP: The equality is true for $n = k + 1$, that is, $1^2 + 2^2 + 3^2 + \dots + k^2 + (k + 1)^2 = ???$

The Proof (of the Inductive Step):

Note: We start with the Left Side of the RTP, and by logical inferences and algebraic manipulations show that it is equal to the Right Side of the RTP.

$$\underbrace{1^2 + 2^2 + 3^2 + \dots + k^2 + (k + 1)^2}_{\text{L.S. of RTP}} \stackrel{\text{Based on the Given}}{=} \underbrace{??? + (k + 1)^2}_{???} = \underbrace{???}_{???} = \underbrace{???}_{???} = \underbrace{???}_{???} = \underbrace{???}_{\text{R.S. of RTP}}$$

$$\Rightarrow 1^2 + 2^2 + 3^2 + \dots + k^2 + (k + 1)^2 = \frac{(k + 1) \cdot (k + 2) \cdot (2k + 3)}{6} \quad \text{Q.E.D. (the mini-proof)}$$

We conclude by Mathematical Induction that: $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$, $\forall n \in N$.

Q.E.D.

Part 3:

1. Prove that $1+2+4+\cdots+2^n = 2^{n+1}-1, \forall n \in \mathbb{N}$
2. Prove that $\forall n \in \mathbb{N}, 10^n - 1$ is divisible by 9.
3. Prove that $2^n > n, \forall n \in \mathbb{N}$
4. The first four terms of a sequence are given as: $a_1 = 3, a_2 = 3^2, a_3 = 3^3, a_4 = 3^4$.
 - 4.1 What is a_5 ? Can you know for sure?
 - 4.2 Can you find a rule for a_n ? What is a_5 according to this rule?
 - 4.3 Can you find another rule, different than the one you found in 4.2? What is a_5 according to this rule?
 - 4.4 Based on your responses to 4.2 and 4.3, what can you say about the sequence a_n ? Is it well defined?