3: SPECIAL FUNCTIONS

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1. Introduction

If you understand exponentials, the key to many of the secrets of the Universe is in your hand.

Carl Sagan, Billions and Billions

Surprisingly, in spite of the abundant data to the contrary, many people believe that human population grows exponentially. It probably never has and probably never will

Joel E. Cohen, How Many People Can The Earth Support?

We now understand derivatives and continuity, so we will soon be able to apply our knowledge to various problems. Before moving onto these applications, we are going to expand our vocabulary of functions. To borrow an analogy from Percy Deift, just as language only requires a few sounds, the applications of mathematics typically only involve combinations of a few special functions. So, we will now discuss in detail a few of the functions that arise most often. Particular attention will be given to the exponential and logarithmic functions, since they appear to be the most elemental.

We begin with the exponential function. We essentially know how to define $2^{1/2}$, $4^{1/3}$, and any positive number to some rational power. However, how do we define 2^{π}? The textbook defines 2^{π} and the exponential function by using a limiting argument. That is, we define 2^{π} as the the limit of the following sequence of rational exponents: $2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, \ldots$. Assuming that this limit exists, properties of the exponential function are then derived.

We will begin by defining the exponential function in a different way, and we will try to include all details about the existence of limits. The key observation is that, once the natural logarithm is defined, we can then easily define the exponential of any number, as we will see in Definition 2.9. For example, in order to define $(2.354)^x$ for $x \in \mathbb{R}$, we just need to carefully define e^x and the natural logarithm.

2. An Alternative Treatment of the Exponential Function

Definition 2.1. Let $n \in \mathbb{Z}$. We define the constant e by the following formula

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

In order for this definition to make sense, we need to know that this limit exists.

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Proof that the limit in Definition 2.1 exists. Let $n \in \mathbb{Z}$, $n > 0$. Using the binomial theorem, we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{j=0}^{n}\left(\frac{1}{n}\right)^j \frac{n!}{j!(n-j)!}\\ &= 1+\frac{1}{n}n+\frac{1}{n^2}\frac{n(n-1)}{2!}+\frac{1}{n^3}\frac{n(n-1)(n-2)}{3!}+\cdots+\frac{1}{n^n}\\ &= 1+1+\frac{n-1}{2n}+\frac{(n-1)(n-2)}{n^2 3!}+\cdots+\frac{1}{n^n}.\end{aligned}$$

Note that, for $j \in \mathbb{Z}$, $j \geq 0$, we have $2^j \leq (j+1)!$, since

$$2^j = \underbrace{2\cdot 2\cdots 2}_{j\text{ times}} \leq 2\cdot 3\cdot 4\cdots(j+1) = (j+1)!$$

So, for $j \in \mathbb{Z}$, $j \geq 1$, we have $1/(j!) \leq 2^{-j+1}$. Therefore,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{j=0}^{n}\left(\frac{1}{n}\right)^j \frac{n!}{j!(n-j)!} = \sum_{j=0}^{n}\frac{1}{n^j}\frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}\\ &\leq \sum_{j=0}^{n}\frac{1}{n^j}\frac{n^j}{j!} = \sum_{j=0}^{n}\frac{1}{j!} = 1+\sum_{j=1}^{n}\frac{1}{j!} \leq 1+\sum_{j=1}^{n}2^{-j+1} = 1+\sum_{j=0}^{n-1}2^{-j} \leq 1+2 = 3.\end{aligned}$$

Here we summed the last geometric series. Recall that if $r < 1$, and $S = 1+r+\cdots+r^n$, then $rS = r+r^2+\cdots+r^{n+1}$, so $S - rS = 1 - r^{n+1}$, so $S = (1-r^{n+1})/(1-r) \leq 1/(1-r)$. So, letting $r = 1/2$ shows that $\sum_{j=0}^{n-1} 2^{-j} \leq 2$.

We now show that, as n increases, $(1+1/n)^n$ increases. Let $k \in \mathbb{Z}$, $0 \leq k \leq n$. We first claim that

$$\frac{n+1-k}{n+1} \geq \frac{n-k}{n}. \qquad (*)$$

To see this, note that $k \geq 0$ implies $0 \geq -k$, so $n^2+n-nk \geq n^2+n-nk-k$, i.e. $n(n+1-k) \geq (n+1)(n-k)$, proving $(*)$. Now,

$$\begin{aligned}\left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n &= \sum_{j=0}^{n+1}\left(\frac{1}{n+1}\right)^j\frac{(n+1)!}{j!(n+1-j)!} - \sum_{j=0}^{n}\left(\frac{1}{n}\right)^j\frac{n!}{j!(n-j)!}\\ &= \frac{1}{(n+1)^{n+1}} + \sum_{j=0}^{n}\left[\frac{1}{(n+1)^j}\frac{(n+1)!}{(n+1-j)!} - \frac{1}{n^j}\frac{n!}{(n-j)!}\right]\frac{1}{j!}\\ &= \frac{1}{(n+1)^{n+1}} + \sum_{j=0}^{n}\left[\frac{(n+1)n(n-1)\cdots(n+2-j)}{(n+1)^j} - \frac{1}{n^j}\frac{n!}{(n-j)!}\right]\frac{1}{j!}\\ &\geq \frac{1}{(n+1)^{n+1}} + \sum_{j=0}^{n}\left[\frac{n(n-1)\cdots(n+1-j)}{n^j} - \frac{1}{n^j}\frac{n!}{(n-j)!}\right]\frac{1}{j!} \qquad \text{, by } (*)\\ &= \frac{1}{(n+1)^{n+1}} + \sum_{j=0}^{n}\left[\frac{1}{n^j}\frac{n!}{(n-j)!} - \frac{1}{n^j}\frac{n!}{(n-j)!}\right]\frac{1}{j!} = \frac{1}{(n+1)^{n+1}} + \sum_{j=0}^{n}[0]\frac{1}{j!} > 0.\end{aligned}$$

In summary, for $n \in \mathbb{Z}$, $n > 0$, we have $(1+1/(n+1))^{n+1} > (1+1/n)^n$, and $(1+1/n)^n \leq 3$. Therefore, the limit $\lim_{n\to\infty}(1+1/n)^n$ actually exists. $\square$

Proposition 2.2. Let x be a rational number, i.e. let $x \in \mathbb{Q}$. Then

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Remark 2.3. Proposition 2.2 differs from Definition 2.1, since in Definition 2.1 we only consider $x \in \mathbb{Z}$, but in Proposition 2.2 we allow $x \in \mathbb{Q}$. Also, we cannot yet allow $x \in \mathbb{R}$ in Proposition 2.2, since we do not yet have a suitable definition for an irrational exponent.

Proof. Fix $x \in \mathbb{Q}$, $x > 1$. Let $n = n(x) \in \mathbb{Z}$ such that $n \leq x < n+1$. Then $1/n \geq 1/x \geq 1/(n+1)$, so $1+1/n \geq 1+1/x \geq 1+1/(n+1)$. Using this inequality and $n+1 \geq x \geq n$,

$$\left(1+\frac{1}{n}\right)^{n+1} \geq \left(1+\frac{1}{x}\right)^x \geq \left(1+\frac{1}{n+1}\right)^n. \qquad (*)$$

We now wish to apply the Squeeze Theorem. As $x \to \infty$, $n \to \infty$, so

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right) = e\cdot 1 = e.$$

Similarly,

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{n+1}\right)^n &= \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n \\ &= \left(\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{-1}\right) = e\cdot 1 = e.\end{aligned}$$

Finally, applying the Squeeze Theorem to $(*)$, we see that $e = \lim_{x\to\infty}(1+1/x)^x$. $\square$

Now that we know the limit in Proposition 2.2 exists, we can define the exponential function for any $x \in \mathbb{R}$.

Definition 2.4. (**The Exponential Function**) Let $x \in \mathbb{R}$ and let $y \in \mathbb{Q}$. Then e^x is defined by the following formula

$$e^x = \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y.$$

Proof that the limit of Definition 2.4 exists. By repeating the proofs of existence for Definition 2.1 and Proposition 2.2, the limit of Definition 2.4 exists for $x \geq 0$. We therefore show that the limit of Definition 2.4 exists also exists for negative values. Let $x \geq 0$. Then

$$\left(1-\frac{x}{y}\right)^y = \frac{\left(1-\frac{x}{y}\right)^y\left(1+\frac{x}{y}\right)^y}{\left(1+\frac{x}{y}\right)^y} = \frac{\left(1-\frac{x^2}{y^2}\right)^y}{\left(1+\frac{x}{y}\right)^y}. \qquad (\ddagger)$$

Now,

$$\left(1-\frac{x^2}{n^2}\right)^n = \left(\sum_{j=0}^{n}\frac{(-1)^j x^{2j}}{n^{2j}}\frac{n!}{j!(n-j)!}\right) = \left(1+\sum_{j=1}^{n}\frac{(-1)^j x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right).$$

So,

$$\lim_{n\to\infty}\left(1-\sum_{j=1}^{n}\frac{x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right)$$
$$\le \lim_{n\to\infty}\left(1+\sum_{j=1}^{n}\frac{(-1)^j x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right)$$
$$\le \lim_{n\to\infty}\left(1+\sum_{j=1}^{n}\frac{x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right). \qquad (*)$$

We now want to apply the Squeeze Theorem. Let $k\in\mathbb{Z}$, $k>1$, and assume that $x>1$. If $n>x^{2k}$, then $n^j>x^{2kj}$, so $n^{-j}<x^{-2kj}$, so $x^{2j}/n^j<x^{2j(1-k)}$, so $-x^{2j}/n^j>-x^{2j(1-k)}$. Then

$$\left(1-\sum_{j=1}^{n}\frac{x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right)\ge\left(1-\sum_{j=1}^{n}\frac{x^{2j}}{n^j}\frac{1}{j!}\right)\ge\left(1-\sum_{j=1}^{n}x^{2j(1-k)}\frac{1}{j!}\right)$$
$$\ge\left(1-x^{1-k}\sum_{j=1}^{n}\frac{1}{j!}\right)\ge 1-3x^{1-k}.$$

Similarly,

$$\left(1+\sum_{j=1}^{n}\frac{x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right)\le\left(1+\sum_{j=1}^{n}\frac{x^{2j}}{n^j}\frac{1}{j!}\right)\le\left(1+\sum_{j=1}^{n}x^{2j(1-k)}\frac{1}{j!}\right)$$
$$\le\left(1+x^{1-k}\sum_{j=1}^{n}\frac{1}{j!}\right)\le 1+3x^{1-k}.$$

Now, as $n\to\infty$, we can take $k\to\infty$, since k can satisfy $n>x^{2k}$. So, $(*)$ becomes

$$\lim_{k\to\infty}(1-3x^{1-k})\le\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n\le\lim_{k\to\infty}(1+3x^{1-k}).$$

Since $x^{1-k}\to 0$ as $k\to\infty$, we conclude from the Squeeze Theorem that $\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n = 1$. So, by (‡),

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n=\frac{\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n}{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}=\frac{1}{e^x}.$$

Then, by repeating the proof of Proposition 2.2, the following limit exists: $\lim_{y\to\infty}(1-x/y)^y$, $y\in\mathbb{Q}$.

Finally, we get the same conclusion for $0\le x\le 1$, since then

$$\left(1-\sum_{j=1}^{n}\frac{x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right)\ge\left(1-\sum_{j=1}^{n}\frac{x^{2j}}{n^j}\frac{1}{j!}\right)\ge\left(1-\sum_{j=1}^{n}\frac{1}{n^j}\frac{1}{j!}\right)$$
$$=\left(1-\frac{1}{n}-\sum_{j=2}^{n}\frac{1}{n^j}\frac{1}{j!}\right)\ge 1-\frac{1}{n}-\frac{1}{n}.$$

Similarly,

$$\left(1+\sum_{j=1}^{n}\frac{x^{2j}}{n^{2j}}\frac{n(n-1)\cdots(n-j+1)}{j!}\right) \leq \left(1+\sum_{j=1}^{n}\frac{x^{2j}}{n^{j}}\frac{1}{j!}\right) \leq \left(1+\sum_{j=1}^{n}\frac{1}{n^{j}}\frac{1}{j!}\right)$$
$$\leq \left(1+\frac{1}{n}+\sum_{j=2}^{n}\frac{1}{n^{j}}\frac{1}{j!}\right) \leq 1+\frac{1}{n}+\frac{1}{n}.$$

Then $(*)$ becomes $\lim_{n\to\infty}(1-2/n) \leq \lim_{n\to\infty}\left(1-x^2/n^2\right)^n \leq \lim_{n\to\infty}(1+2/n)$, so

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = \frac{\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n}{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n} = \frac{1}{e^x}.$$

Then, by repeating the proof of Proposition 2.2, the following limit exists: $\lim_{y\to\infty}\left(1-x/y\right)^y$, $y \in \mathbb{Q}$. □

Remark 2.5. From the Definition 2.4 for $x \geq 0$, we see that $e^x \geq 1$ and $e^0 = 1$. Since $e^x e^{-x} = 1$, we see that $e^{-x} > 0$. So, $e^x > 0$ for all $x \in \mathbb{R}$. Also, since $e^x = \lim_{n\to\infty}(1+x/n)^n = \lim_{n\to\infty}\sum_{j=0}^{n}\frac{x^j}{n^j}\frac{n!}{j!(n-j)!} \geq 1+x$, we see that

$$\lim_{x\to\infty} e^x = \infty.$$

Since $e^x e^{-x} = 1$, we have $e^{-x} = 1/e^x$, so

$$\lim_{x\to-\infty} e^x = \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} 1/e^x = 1/(\lim_{x\to\infty} e^x) = 0.$$

By modifying the proof after Definition 2.1, we see that e^x is a continuous, strictly increasing function. So, by the Intermediate Value Theorem, the range of the exponential function is $(0,\infty)$. Now, e^x is continuous and strictly increasing with domain $\mathbb{R}$ and range $(0,\infty)$. So, for any $y \in (0,\infty)$, there exists a unique $x \in \mathbb{R}$ such that $e^x = y$. If there were $x, x' \in \mathbb{R}$ with $x \neq x'$ and $e^x = e^{x'}$, then it would be impossible to define the inverse function of e^x. There would be an ambiguity, since the inverse function evaluated at e^x could be set to x or x'. Since there exists a unique $x \in \mathbb{R}$ such that $e^x = y$, we can actually define the inverse function of the exponential.

Definition 2.6. (**The Natural Logarithm**) Let $x > 0$. Then the natural logarithm $\log(x)$ is defined to be the inverse of e^y. That is, $y = \log(x)$ if and only if $e^y = x$.

Remark 2.7. In the notation of the book, we have $\log(x) = \ln(x)$.

Proposition 2.8. For $x, y \in \mathbb{R}$, $e^{x+y} = e^x e^y$.

Proof. Let $x, y, w, q, r \in \mathbb{Q}$. Then

$$\begin{aligned}
e^{x+y} &= \lim_{w\to\infty}\left(1+\frac{x+y}{w}\right)^{w} = \left(\lim_{w\to\infty}\left(1+\frac{x+y}{w}\right)^{w/(x+y)}\right)^{x+y} \\
&= \left(\lim_{w\to\infty}\left(1+\frac{x+y}{w}\right)^{w/(x+y)}\right)^{x}\left(\lim_{w\to\infty}\left(1+\frac{x+y}{w}\right)^{w/(x+y)}\right)^{y} \\
&= \left(\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^{q}\right)^{x}\left(\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^{q}\right)^{y} = \left(\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^{qx}\right)\left(\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^{qy}\right) \\
&= \left(\lim_{r\to\infty}\left(1+\frac{x}{r}\right)^{r}\right)\left(\lim_{r\to\infty}\left(1+\frac{y}{r}\right)^{r}\right) = e^{x}e^{y}.
\end{aligned}$$

Since the exponential function is continuous, and the identity $e^x e^y = e^{x+y}$ holds for rational x, y, we conclude that the identity $e^x e^y = e^{x+y}$ therefore holds for all $x, y \in \mathbb{R}$. □

Definition 2.9. Let $x, y \in \mathbb{R}$, $y > 0$. Then the exponential function y^x is defined by

$$y^x = e^{x\log(y)}.$$

Definition 2.10. Let $x, y > 0$. Then $\log_y(x)$ is defined by

$$\log_y(x) = \frac{\log(x)}{\log(y)}.$$

Proposition 2.11. Let $x, y > 0$. Then $\log(xy) = \log(x) + \log(y)$. Also, $\log(x^y) = y\log(x)$, and $\log(x/y) = \log(x) - \log(y)$.

Proof. For $z > 0$, $e^{\log(z)} = z$, since the exponential and logarithmic functions are inverses. Using this property and also Proposition 2.8,

$$e^{\log(xy)} = xy = e^{\log(x)}e^{\log(y)} = e^{\log(x)+\log(y)}.$$

Taking the logarithm of both sides and applying the identity $\log(e^w) = w$ for $w \in \mathbb{R}$, we get $\log(xy) = \log(x) + \log(y)$. For the third property, we use the following identity.

$$e^{\log(x/y)} = x/y = e^{\log(x)}e^{-\log(y)} = e^{\log(x)-\log(y)}.$$

For the second property, use Definition 2.9 to get

$$\log(x^y) = \log(e^{y\log(x)}) = y\log(x).$$

□

In order to differentiate exponential functions, we will need the following proposition.

Proposition 2.12. Let $f\colon (0,\infty) \to \mathbb{R}$ be a differentiable function such that: for all $a \in \mathbb{R}$, there exists a unique $x \in (0,\infty)$ such that $f(x) = a$. Assume also that $f'(x) \neq 0$ for all $x \in (0,\infty)$, and that f is strictly increasing. Then the inverse function $f^{-1}\colon \mathbb{R} \to (0,\infty)$ exists and is differentiable. Moreover,

$$\frac{d}{da}(f^{-1})(a) = \frac{1}{f'(f^{-1}(a))}.$$

Proof. Let $c, d \in (0, \infty)$ with $c < d$. We first show that f^{-1} is continuous. By Lemma 6.3 of the first set of notes, it suffices to show that $f(c, d)$ is an open interval. By the Intermediate Value Theorem, $f(c, d)$ contains $(f(c), f(d))$. Since f is increasing, $f(c, d)$ is contained in $(f(c), f(d))$. Combining these two containments, we conclude that $f(c, d) = (f(c), f(d))$, so $f(c, d)$ is indeed an open interval. And Lemma 6.3 of the first set of notes shows that f^{-1} is therefore continuous.

Now, as $b \to a$, the continuity of f^{-1} shows that $f^{-1}(b) \to f^{-1}(a)$. That is, as $b \to a$ we have $d \to c$. So,

$$\lim_{b\to a} \frac{f^{-1}(a) - f^{-1}(b)}{a - b} = \lim_{b\to a} \frac{1}{\frac{a-b}{f^{-1}(a)-f^{-1}(b)}} = \lim_{d\to c} \frac{1}{\frac{f(c)-f(d)}{c-d}} = \frac{1}{f'(c)} = \frac{1}{f'(f^{-1}(a))}.$$

□

Now that we have our definitions and key properties accounted for, we can finally prove some things about the derivatives of exponential and logarithmic functions.

Proposition 2.13. Let $x, y \in \mathbb{R}$, $y > 0$.

(a) $(d/dx)(e^x) = e^x$.
(b) $(d/dx)(y^x) = y^x \log(y)$.
(c) $(d/dy)(\log(y)) = 1/y$.

Proof of (c). We first prove that

$$\lim_{u\to 0}(1 + u)^{1/u} = e. \qquad (*)$$

To see this, first apply Proposition 2.2 to get

$$\lim_{u\to 0^+}(1 + u)^{1/u} = \lim_{x\to\infty}(1 + 1/x)^x = e.$$

Then, apply Definition 2.4 to get

$$\lim_{u\to 0^-}(1 + u)^{1/u} = \lim_{x\to-\infty}(1 + 1/x)^x = 1/(\lim_{x\to\infty}(1 + (-1)/x)^x) = 1/(e^{-1}) = e.$$

Since both the left and right limits agree, statement $(*)$ is proven.

We now prove that $\log\colon (0, \infty) \to \mathbb{R}$ is continuous. Let $f(x) = e^x$ and let $c, d \in (-\infty, \infty)$ with $c < d$. By Lemma 6.3 of the first set of notes, it suffices to show that $f(c, d)$ is an open interval. Since f is continuous by Remark 2.5, the Intermediate Value Theorem says $f(c, d)$ contains $(f(c), f(d))$. Since f is increasing by Remark 2.5, we get that $f(c, d)$ is contained in $(f(c), f(d))$. Combining these two containments, we conclude that $f(c, d) = (f(c), f(d))$, so $f(c, d)$ is indeed an open interval. And Lemma 6.3 of the first set of notes shows that $\log = f^{-1}$ is therefore continuous.

Now,

$$\begin{aligned}\lim_{h\to 0} \frac{\log(y + h) - \log(y)}{h} &= \lim_{h\to 0}\left[\frac{1}{h}\log\left(\frac{y + h}{y}\right)\right] &&\text{, by Proposition 2.11}\\ &= \frac{1}{y}\lim_{h\to 0}\left[\frac{y}{h}\log\left(1 + \frac{h}{y}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{y}\lim_{h\to 0}\log\left[\left(1+\frac{h}{y}\right)^{y/h}\right] &&\text{, by Proposition 2.11}\\
&= \frac{1}{y}\lim_{u\to 0}\log\left[(1+u)^{1/u}\right]\\
&= \frac{1}{y}\log\left[\lim_{u\to 0}(1+u)^{1/u}\right] &&\text{, since log is continuous}\\
&= \frac{1}{y} \quad \text{, by } (*)
\end{aligned}
$$

□

Proof of (a). Let $f(y) = \log(y)$. As described in Remark 2.5, the exponential function is increasing. That is, $a > b$ if and only if $e^a > e^b$. Applying this to $a = \log(x)$ and $b = \log(y)$, we have $\log(x) > \log(y)$ if and only if $x > y$, i.e. $f(y)$ is also increasing. From part (a), f is differentiable and $f'(y) \neq 0$ for all $y > 0$. Therefore, we can apply Proposition 2.12, to get

$$\frac{d}{dx}(e^x) = \frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} = f^{-1}(x) = e^x.$$

□

Proof of (b). Let $f(x) = y^x = e^{x\log(y)}$. From the Chain Rule and part (b), $f'(x) = e^{x\log(y)}\log(y) = y^x\log(y)$. □

We can now prove the power law for derivatives

Proposition 2.14. Let $x > 0, y \in \mathbb{R}$. Let $f(x) = x^y$. Then $f'(x) = yx^{y-1}$.

Proof. From the Chain Rule and Proposition 2.13(c),

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}[\log(f(x))] = \frac{d}{dx}[y\log(x)] = \frac{y}{x}.$$

So,

$$f'(x) = \frac{y}{x}f(x) = yx^{y-1}.$$

□

3. A Proof of L'Hôpital's Rule

Theorem 3.1. ***(Extreme Value Theorem)*** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then f achieves its minimum and maximum values. More specifically, there exist $c, d \in [a,b]$ such that: for all $x \in [a,b]$, $f(c) \le f(x) \le f(d)$.*

Proof. Let $g(t) = f(t(b-a) + (1-t)a)$, so that $g\colon [0,1] \to \mathbb{R}$ is continuous. We will show that g achieves its maximum and minimum values, implying that f achieves its maximum and minimum values. First of all, by the Intermediate Value Theorem, we know that $g[0,1]$ is an interval. We just do not know whether or not $g[0,1]$ contains the endpoints of that interval. Suppose the endpoints of this interval are y, z with $y < z$. In particular, suppose $g[0,1] \supseteq (y,z)$. Now, let $x_1, x_2, x_3, \ldots \in [0,1]$ such that $g(x_i) \to z$ as $i \to \infty$. Since $[0,1] = [0,1/2] \cup [1/2,1]$, we can find $x_{j_1}, x_{j_2}, \ldots \in [0,1/2]$ or $x_{j_1}, x_{j_2}, \ldots \in [1/2,1]$ such that $g(x_{j_i}) \to z$ as $i \to \infty$. Suppose we have $x_{j_1}, x_{j_2}, \ldots \in [0,1/2]$ such that $g(x_{j_i}) \to z$

as $i \to \infty$. Since $[0, 1/2] = [0, 1/4] \cup [1/4, 1/2]$, we can find $x_{j_{k_1}}, x_{j_{k_2}}, \ldots \in [0, 1/4]$ or $x_{j_{k_1}}, x_{j_{k_2}}, \ldots \in [1/4, 1/2]$ such that $g(x_{j_{k_i}}) \to z$ as $i \to \infty$. By repeating this procedure an infinite number of times, there exist $d_1, d_2, \ldots \in [0, 1]$ and there exists $d \in [0, 1]$ such that $d_i \to d$ as $i \to \infty$ and $g(d_i) \to z$ as $i \to \infty$. Since g is continuous, $g(d_i) \to g(d)$ as $i \to \infty$. Combining these two statements, we see that $g(d) = z$. By repeating this argument, we can also find c such that $g(c) = y$. In conclusion, $g[0, 1] = [y, z]$, as desired. □

Remark 3.2. This Theorem is not true if f is discontinuous. For example, consider

$$f(x) = \begin{cases} x+1 & , x < 0 \\ x-1 & , x > 0 \\ 0 & , x = 1 \end{cases}.$$

Consider f on the interval $[-1, 1]$. Then $f(-1) = 0$, $f(1) = 0$, $f(x) < 1$ for $x \in [-1, 1]$ and $f(x) > -1$ for $x \in [-1, 1]$. However, $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = -1$. And by the Intermediate Value Theorem, $f[-1, 0) = [0, 1)$ and $f(0, 1] = (-1, 0]$. So, $f[-1, 1] = (-1, 1)$. And we cannot find $c, d \in [-1, 1]$ with $f(c) = -1$ and $f(d) = 1$.

Theorem 3.3. ***(Rolle's Theorem)*** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous function that is differentiable on (a, b) with $f(a) = f(b) = 0$. Then there exists c with $c \in (a, b)$ and $f'(c) = 0$.*

Proof. If $f = 0$ everywhere there is nothing to prove, so we assume otherwise. Without loss of generality, assume that there exists a point $d \in [a, b]$ such that $f(d) > 0$. By the Extreme Value Theorem, Theorem 3.1, let $c \in [a, b]$ be such that, for all $x \in [a, b]$ $f(c) \geq f(x)$. In particular, $f(c) \geq f(d) > 0$. Also, since $f(a) = f(b) = 0$, we must have $c \in (a, b)$. Let $h > 0$ so that h is less than the minimum of $|c - a|/2$ and $|c - b|/2$. By our choice of c, we have $f(c + h) \leq f(c)$ and $f(c - h) \leq f(c)$. Therefore, by the Squeeze Theorem,

$$0 \geq \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} = f'(c) = \lim_{h \to 0^+} \frac{f(c) - f(c-h)}{h} \geq 0.$$

We conclude that $f'(c) = 0$. □

Theorem 3.4. ***(Mean Value Theorem)*** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous function that is differentiable on (a, b). Then there exists c with $c \in (a, b)$ and*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $g(x) = f(x) - f(a) - (x - a)((f(b) - f(a))/(b - a))$. Then $g(a) = g(b) = 0$. So, by applying Rolle's Theorem, Theorem 3.3, to g, there exists $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

That is, $f'(c) = (f(b) - f(a))/(b - a)$. □

Theorem 3.5. ***(Cauchy's Mean Value Theorem)*** *Let $f, g\colon [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b) with $g'(x) \neq 0$ for $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. First, note that $g(b) - g(a) \neq 0$, because if $g(b) - g(a) = 0$, then the Mean Value Theorem would imply that there exists $d \in (a, b)$ such that $g'(d) = 0$. But we have assumed that $g'(x) \neq 0$ for $x \in (a, b)$. Now, let

$$h(x) = f(x) - f(a) - (g(x) - g(a))\frac{f(b) - f(a)}{g(b) - g(a)}.$$

By applying Rolle's Theorem, Thm. 3.3, to h, there exists $c \in (a, b)$ such that

$$0 = h'(c) = f'(c) - g'(c)\frac{f(b) - f(a)}{g(b) - g(a)}.$$

That is, $f'(c)/g'(c) = (f(b) - f(a))/(g(b) - g(a))$. □

Theorem 3.6. ***(L'Hôpital's Rule)*** *Let $c < a < d$. $f, g\colon (c, a) \cup (a, d) \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ for $x \in (c, a) \cup (a, d)$. Alternately, let $a = \infty$, let $f, g\colon (c, a) \to \mathbb{R}$ be differentiable with $g'(x) \neq 0$ for $x \in (c, a)$. Also, assume that one of the two following possibilities occurs*

(1) $\lim_{x \to a} f(x) = 0$ *and* $\lim_{x \to a} g(x) = 0$.
(2) $\lim_{x \to a} |f(x)| = \infty$ *and* $\lim_{x \to a} |g(x)| = \infty$.

Moreover, assume that $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. Let $c < x < y < a$. If there exists $x \in (c, a)$ such that $g(x) = 0$, then all other $x' \in (x, a)$ must satisfy $g(x') \neq 0$. To see this, assume that $x' \in (x, a)$ satisfies $g(x') = 0$. Then we would have $g(x) - g(x') = 0$, which would imply the existence of $r \in (x, a)$ such that $g'(r) = 0$. However, we assumed that $g'(x) \neq 0$ for $x \in (c, a)$. Therefore, by choosing c to be larger if necessary, we may assume that $c < x < y < a$ and $g(x) \neq 0$ for all $x \in (c, a)$.

By Cauchy's Mean Value Theorem, Theorem 3.5, there exists $z \in (x, y)$ such that

$$\frac{f'(z)}{g'(z)} = \frac{f(x) - f(y)}{g(x) - g(y)}. \qquad (*)$$

Suppose Case (1) occurs. We rewrite $(*)$ as

$$\frac{f'(z)}{g'(z)} = \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)}}{1 - \frac{g(y)}{g(x)}}. \qquad (**)$$

Let $y \to a$. Then $g(y) \to 0$ and $f(y) \to 0$ since we are in Case (1), so $(**)$ says

$$\min_{z \in (x,a)} \frac{f'(z)}{g'(z)} \leq \frac{f(x)}{g(x)} \leq \max_{z \in (x,a)} \frac{f'(z)}{g'(z)}.$$

Finally, letting $x \to a$ and using that $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, the Squeeze Theorem implies

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

Case (1) is complete.

Suppose Case (2) occurs. Then

$$\frac{f'(z)}{g'(z)} = \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{\frac{f(x)}{g(y)} - \frac{f(y)}{g(y)}}{\frac{g(x)}{g(y)} - 1}.$$

Let $y \to a$. Then $1/g(y) \to 0$, so

$$\min_{z\in(x,a)} \frac{f'(z)}{g'(z)} \leq \min_{z\in(x,a)} \frac{f(z)}{g(z)} \leq \max_{z\in(x,a)} \frac{f(z)}{g(z)} \leq \max_{z\in(x,a)} \frac{f'(z)}{g'(z)}.$$

Finally, letting $x \to a$ and using that $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, the Squeeze Theorem implies

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$

Case (2) is complete. □

4. Selected Exercises

Exercise 4.1. Let $f(x) = 10^x$. What is $(d/dx)f(x)$?

Exercise 4.2. (page 178 from the textbook) Scientists can determine the age of ancient objects by the method of **radiocarbon dating**. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of Carbon-14, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates Carbon-14 through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of Carbon-14 begins to decrease through radioactive decay. Therefore the level of radioactivity of the carbon must also decay exponentially.

A parchment fragment was discovered that had about 74% as much Carbon-14 radioactivity as does plant material on the earth today. Estimate the age of the parchment.

Exercise 4.3. Suppose I put 100 dollars in my bank account, and the interest rate is 6 percent, compounded annually. Then in one year, I will have $100(1+.06)$ dollars. If the interest is compounded daily, then after one year, I will have $100(1+.06/365)^{365}$ dollars. If the interest is compounded every second, then after one year, I will have $100(1+.06/31536000)^{31536000}$ dollars. Suppose the unit of time over which the interest is computed goes to zero. In this scenario, we say that the interest is computed "continuously." After a year of 6% interest computed continuously, how much money will I have?

Exercise 4.4. Let n be a positive integer. Prove that

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

Exercise 4.5. (**Chemistry 101**) In chemistry class, we learn two basic formulas for chemical kinetics. Here we will investigate these formulas. Let $f(t)$ be the concentration of a chemical in a reaction at time t. Let $k > 0$, let $a > 0$, and assume that $f(0) = a$.

(a) A "first-order" reaction satisfies the following equation

$$\frac{df}{dt} = -kf(t), \qquad f(0) = a \qquad (*)$$

Show that $f(t) = ae^{-kt}$ satisfies $(*)$.

(b) A "second-order" reaction satisfies the following equation

$$\frac{df}{dt} = -k(f(t))^2, \qquad f(0) = a \qquad (**)$$

Show that $f(t) = \frac{1}{a^{-1}+kt}$ satisfies $(**)$.

Exercise 4.6. (**Drug Concentration**) Suppose a human body contains 2 gallons of blood, and a therapeutic drug has 2% concentration in the blood. Suppose we administer IV fluids into the bloodstream at a rate of 1/4 gallon per hour. Assume also that the kidneys filter the drug out of the blood at a rate of 1/4 gallon of blood per hour. Let $y(t)$ be the concentration of the drug, where t is a unit in hours, and suppose $y(0) = .02$. By our assumption, the rate of change of the concentration $y(t)$ is given by $dy/dt = -y(t)/4$. Show that $y(t) = .02e^{-t/4}$ satisfies $y(0) = .02$ and $dy/dt = -y(t)/4$.

Exercise 4.7. (**A speeding ticket?**) Suppose I am driving in a car, and there are police cameras that are stationed at certain mile markers. The first camera spots my license plate at 10 AM. Five miles down the road, the second camera spots my license plate at 10 : 04 AM. If my speed exceeded 74 miles per hour at any particular point in time, I will automatically be issued a ticket in the mail. Will I be issued a ticket?

Exercise 4.8. What is $\lim_{x\to 0^+} x\log(x)$? What is $\lim_{x\to 0^+} x^{1/10}\log(x)$?

Exercise 4.9. (**The shape of hanging cables and chains**) Suppose we have a cable hanging between two poles of equal height. We will derive the shape of the hanging cable. That is, we will find a function $y = f(x)$, with $x = 0$ the midpoint of the cable, such that the cable follows the curve $y = f(x)$. At the outset, we assume that the function $y = f(x)$ is differentiable.

Consider a segment of the cable from $x = 0$ to $x = b > 0$. We consider this segment of cable as a single body. Then there are three distinct forces acting on this segment of cable. At $x = 0$, we assume that the curve $y = f(x)$ has a horizontal tangent. First, there is a tension force T_0 pulling the cable in the negative x direction at $x = 0$, so that this force is tangent to the curve at $x = 0$. Second, there is a tension force T pulling the cable at $x = b$, and this force is also tangent to the curve at $x = b$. Third, the force of gravity of the segment of chain from $x = 0$ to $x = b$ pulls straight down. This third force is denoted $-\rho g s(b)$, where g is the force of gravity, ρ is a constant, and $s(b)$ is the length of the chain from $x = 0$ to $x = b$.

Adding all three forces together, we must get zero, since the cable is hanging in equilibrium. Suppose at $x = b$ that the tangent line to $y = f(x)$ makes an angle θ with the x-axis. Then the sum of forces in the x-direction is $-T_0 + T\cos(\theta)$, and the sum of forces in the y-direction is $-\rho g s(b) + T\sin(\theta)$. So, $T_0 = T\cos(\theta)$ and $\rho g s(b) = T\sin(\theta)$. Since $(df/dx)(b) = \sin(\theta)/\cos(\theta)$, we have

$$\frac{df}{dx}(b) = \frac{\rho g s(b)}{T_0}. \qquad (*)$$

Now, $s(b)$ is the length of $f(x)$ from $x = 0$ to $x = b$. Let $h > 0$, and assume that s is differentiable. From the linear approximation of the derivative, we have $f(b+h) \approx f(b) + hf'(b)$. Consider the right triangle with vertices $(b, f(b))$, $(b+h, f(b+h))$, and $(b+h, f(b))$. The length of the hypotenuse of this triangle is approximately $s(b+h) -$

$s(b)$. Also, from the Pythagorean Theorem, the length of the hypoteuse of this triangle is $\sqrt{h^2 + (f(b+h) - f(b))^2}$. Combining these facts,

$$s(b+h) - s(b) \approx \sqrt{h^2 + (f(b+h) - f(b))^2} \approx \sqrt{h^2 + h^2(f'(b))^2}.$$

So, dividing both sides by $h > 0$ we have $(s(b+h) - s(b))/h \approx \sqrt{1 + (f'(b))^2}$. Letting $h \to 0^+$, and then taking a derivative of $(*)$, we have derived the following equation

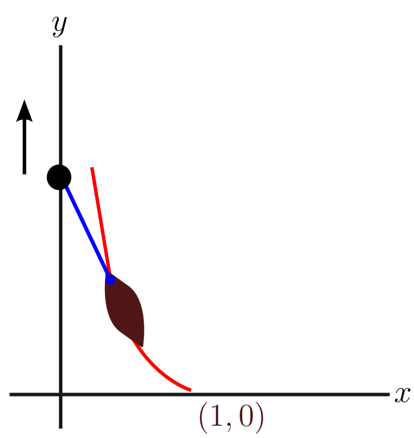
$$\frac{d^2 f}{dx^2}(b) = \frac{\rho g}{T_0}\sqrt{1 + \frac{df}{dx}(b)}. \qquad (**)$$

We can now solve for f. Show that the shape of the cable is described by

$$y = f(x) = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right).$$

The shape of the hanging cable, known as a catenary curve, also appears in architecture, such as the St. Louis Gateway Arch. The idea is to freeze the hanging cable in its position, and then flip it upside down to produce an arch. Then we can repeat the derivation above to see that the forces of the arch are all the same as in the case of the chain (though the signs of the forces are flipped). Also, for a very small segment of the arch, we can essentially neglect the gravitational force exerted on this segment. So, by reviewing the above analysis, the force on any particular point in the arch will be directed along the arch itself. Therefore, the arch is very stable.

5. Selected Problems



Exercise 5.1. (**Towing an unconstrained object**, Thomas' Calculus, p. 544) Suppose I am standing on the shore of a straight river, and I am pulling on a rope of length 1 connected to the front of a canoe. I am walking at a constant speed in the positive y-direction. Suppose the canoe is in the river, and the canoe's front is at a distance x from the river shore. Suppose the initial position of the canoe is $(1, 0)$. As I move in the positive y-direction, the canoe's front is pulled along a curve denoted by $y = f(x)$. If the rope is taut, it will always be tangent to the curve $f(x)$. Consider the right triangle formed by: me on the shore, the

point $(x, f(x))$ and the y-axis. Then the height of this triangle in the y-direction is $\sqrt{1-x^2}$. Therefore,

$$\frac{df}{dx} = -\frac{\sqrt{1-x^2}}{x}$$

Show that $f(x) = \mathrm{sech}^{-1}(x) - \sqrt{1-x^2}$ satisfies $\frac{df}{dx} = -\frac{\sqrt{1-x^2}}{x}$.

Problem 5.2. -(MIT 18.01SC, Final Exam, 3b) What is

$$\lim_{x\to\sqrt{3}} \frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}}?$$

Problem 5.3. What is $\lim_{x\to 0^+} x^x$? (Hint: use the second part of Proposition 2.11.)

Problem 5.4. Let $f(x) = x^x$. What is $(d/dx)f(x)$? (Hint: use Definition 2.10.)

Problem 5.5. In the above Exercises and Problems, and in many applications, we are given a set of equations of the form

$$\frac{df}{dt} = g(f(t)), \quad f(0) = a \qquad (\ddagger)$$

Here $g\colon \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ are given. And we want to find an f that satisfies (‡). Given such a set of equations (‡), do you think it is always possible to find an f satisfying (‡)? If so, this would be very nice, since finding f means that we better understand whatever application gave us (‡). For example, in Exercise 4.5, our mathematical model of the chemical reaction is at least sensible, since the solution f of the equation (‡) exists. If we could not find an f satisfying (‡), we might need to question the correctness of the reaction model that gave us (‡). Or, we would need to create more mathematical tools to understand when we can or cannot find an f satisfying (‡).

It turns out that, for "reasonable" g, the answer to our question is yes. We can find f satisfying (‡). However, we will not learn why this is so in this class. Nevertheless, the answer to this question is extremely important, and it has ramifications for an extremely large number of applications of mathematics to the real world.

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