1: INTRODUCTION

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1. Applications of Calculus

It seemed a limbo of painless patient consciousness through which souls of mathematicians might wander, projecting long slender fabrics from plane to plane of ever rarer and paler twilight, radiating swift eddies to the last verges of a universe ever vaster, farther and more impalpable.

James Joyce, A Portrait of the Artist as a Young Man.

The ability of Calculus to describe the world serves as one of the great triumphs of mathematics. Below, we will briefly describe some of the applications of Calculus. Some of these applications will be discussed in the second half of this course, and others may be found in your future endeavors.

In **economics**, there are many quantities that one would like to optimize. For example, one may want to minimize the cost of producing certain goods. In general, one often wants to do something in the best way possible. We will describe one way of doing something in the best way in the section on **optimization**. Many of the ideas that appear in this calculus class reappear in **stochastic calculus**. In stochastic calculus, one wants to better understand stock prices, which are modeled as random functions.

In **physics**, many models of the real world use solutions of **differential equations**. These equations involve the slopes and shape of functions, and their solutions describe the behavior of many physical systems. For example, the famous Navier-Stokes equations of fluid dynamics are expressed in this language. Solutions of the Navier-Stokes equations show us how water behaves, though these equations really just state Newton's second law. Also, **Einstein's Theory of General Relativity** uses a version of Calculus, though geometry is needed here as well.

In **mathematics** itself, the fundamental concepts of Calculus reappear in many places, some of which have already been described above. Also, Calculus serves as the foundation of **probability**, which itself serves as the foundation of **statistics**. For example, to prove that a large number of numerical data samples have the distribution of a bell curve, one can use tools from Calculus. As another example, we can repeat Calculus for a single real variable by using a single complex variable, and we get the beautiful subject of **Complex analysis**.

For a complete understanding of **biology**, you need to understand Calculus. For example, suppose someone is given an intravenous drug which is administered at a certain rate. If we know the volume of fluid in the body, the concentration of the administered drug, and the rate of flow of the drug into the body, then we can use Calculus to model the concentration of drug in the person's body. These calculations are done using differential equations. If one wants to fully understand how MRIs and CT-scans work, one needs **multivariable calculus**.

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Many of the concepts in multivariable calculus originate in single variable calculus, though there are some new things that are needed.

Using differential equations, one can derive some of the equations for chemical concentrations that were used in our high school **chemistry** classes. Also, the ideas of Calculus are used in more advanced chemistry subjects, including **quantum mechanics**.

Even though **computer science** often deals with discrete problems, many of the ideas of Calculus arise in computer science, and sometimes these ideas arise in unexpected ways. For example, the methods used to encode music onto CDs and MP3s use Calculus, with some additional tools from **Fourier analysis**. These same tools compress image and video data for JPEGs and MPEGs, respectively.

The acts of thinking rigorously and using **logic** in our reasoning should become common in this course. We want to transmit one of the great intellectual achievements of humanity, just as we pass down great literature, art and philosophy to future generations.

2. A Brief History of Calculus

The rudiments of Calculus can be traced through many ancient cultures including those of Greece, China and India. Calculus in its modern form is generally attributed to Newton and Leibniz in the late 1600s. Newton was mainly motivated by applying Calculus to physics. However, Leibniz invented most of the notation we use today. Even though this topics is now taught in high school and college, it is still over 300 years old. In this course, perhaps we will try to give you glimpses of what happened in the ensuing centuries.

3. The Calculus Paradigm

Mathematics before Calculus usually involves algebra, trigonometry and some planar geometry. The concepts in Calculus are very different from concepts that are learned before, so the following "paradoxes" are meant to help in understanding the new concepts. These paradoxes are known as Zeno's Paradoxes.

Paradox 3.1. In a footrace, suppose a slower runner is in front of a quicker runner. When the quicker runner reaches any point in the race, the slower runner was already at that point in the past. Therefore, the quicker runner can never overtake the slower runner.

Paradox 3.2. Suppose I want to walk through a doorway, and I am standing a meter away from the door. At some point I am at a half meter away from the door, then at another point I am at a quarter of a meter away, then at another point I am at an eighth a meter away, and so on. Therefore, I can never make it through the doorway.

Paradox 3.3. Suppose I shoot at arrow at a target. At any given moment, the arrow occupies a fixed position in space. However, in order for an object to move, it cannot sit in one place. So, the arrow must have no motion at all. The arrow is motionless.

The first two paradoxes are somewhat similar. In the first paradox, we know from empirical observation that a quicker runner can overtake a slower runner. And in order to resolve the paradox, we need to note that the total time that the quicker runner remains behind the slower runner is finite. Similarly, in the second paradox, I know that it only takes a finite amount of time to pass through a door. Paradox 3.2 seems to occur since I am subdividing the one meter that I travel into an infinite number of smaller steps. Zeno seems to object, saying that an infinite number of subdivisions cannot occur.

In mathematical terms, Zeno objects to the assertion $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$, because there does not appear to be a rigorous way to think about an infinite sum of numbers. By using a limit, we can actually speak rigorously about an infinite sum of numbers, thereby resolving the paradox.

The third paradox is a bit different from the other two. Zeno is really questioning the meaning of an instant of time. How can we rigorously discuss the instantaneous speed of an object? In Chapter 2 of the textbook, we will use limits to define derivatives, and these derivatives give a rigorous meaning to instantaneous velocity, thereby resolving the paradox.

4. The Notion of a Limit

Let $f\colon \mathbb{R}\to\mathbb{R}$, and let $x,a,L\in\mathbb{R}$. Intuitively, we say that f has limit L as x approaches a, if $f(x)$ gets closer and closer to L as x gets closer and closer to a. More rigorously,

Definition 4.1. Fix $a\in\mathbb{R}$ and let x be a variable. We write $\lim_{x\to a} f(x)=L$ if the following occurs.

$$\begin{gathered}\text{For all } \varepsilon>0,\text{ there exists } \delta(\varepsilon)>0,\text{ such that:}\\ \text{if } 0<|x-a|<\delta(\varepsilon),\text{ then } |f(x)-L|<\varepsilon.\end{gathered}\tag{1}$$

So, no matter how close I want f to be to L, I can always choose a small region around a that is so small such that f is really close to L.

The limit may at first look a bit silly[1], since for a polynomial, we can just plug in the function value and the result agrees with the limit. So why are we defining a limit anyway? First of all, we need to define limits to define derivatives, and derivatives are one of the extremely important concepts in Calculus. Second of all, there are some subtleties to the definition that may not yet be apparent.

For example, $\lim_{x\to a} f(x)$ does not depend on the value of the function f at a. To see this, note that the definition of a limit only states a condition about $0<|x-a|$. As a more important issue, the limit of a function may not always exist. This issue is important because we will see that the derivatives of some functions may not exist. So, there may be no reasonable way to talk about the instantaneous speed of certain trajectories.

If the limit of a function does not exist, we sometimes write DNE. Here are three important examples where a limit does not exist. Since we would like to think about the rate of change of many functions, it is important to understand the cases when we cannot talk about the rate of change of certain functions, i.e. when certain limits fail to exist. In order to better understand the definition of the limit (1), we will apply this definition to the examples below.

Example 4.2. (**A jump discontinuity**) Define $H\colon\mathbb{R}\to\mathbb{R}$ by the formula

$$H(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$$

Claim: $\lim_{x\to0} H(x)$ DNE, $\lim_{x\to0^-} H(x)=0$ and $\lim_{x\to0^+} H(x)=1$.

[1]Historically, the definition (1) was invented over a century after Newton and Leibniz invented Calculus. The approach of Newton and Leibniz was not quite rigorous by modern standards.

Proof of Claim. Let $x < 0$, and let $\varepsilon > 0$. Using (1), I need to find δ such that: $-\delta < x < 0$ implies $|H(x) - 0| < \varepsilon$. But I can just choose $\delta = 1$. Since $x < 0$, $H(x) = 0$, so $|H(x) - 0| = 0 < \varepsilon$. We conclude that $\lim_{x\to 0^-} H(x) = 0$.

Now, let $x > 0$, and let $\varepsilon > 0$. I need to find δ such that: $0 < x < \delta$ implies $|H(x) - 1| < \varepsilon$. I can just choose $\delta = 1$ again. Since $x > 0$, $H(x) = 1$, so $|H(x) - 1| = 0 < \varepsilon$. We conclude that $\lim_{x\to 0^+} H(x) = 1$.

Now, if $\lim_{x\to 0} H(x)$ exists, then $\lim_{x\to 0^+} H(x) = \lim_{x\to 0^-} H(x)$. Taking the contrapositive: if $\lim_{x\to 0^+} H(x) \neq \lim_{x\to 0^-} H(x)$, then $\lim_{x\to 0} H(x)$ does not exist. Since $\lim_{x\to 0^+} H(x) \neq \lim_{x\to 0^-} H(x)$, we conclude that $\lim_{x\to 0} H(x)$ does not exist. □

Example 4.3. (**A singularity**) Define $f\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Claim: $\lim_{x\to 0} f(x)$ DNE, $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$.

Proof of Claim. We will first show that $\lim_{x\to 0} f(x)$ does not exist. We argue by contradiction. Suppose $L \in \mathbb{R}$ and $\lim_{x\to 0} f(x) = L$. We now apply the definition (1). Let $\varepsilon = 1$. Then there exists $\delta > 0$ such that: $0 < |x| < \delta$ implies $|f(x) - L| < 1$. Let x' be the minimum of $\delta/2$ and $1/(|L| + 2)$. Since $0 < x' \leq 1/(|L| + 2)$, we have $1/x' \geq |L| + 2$. Also, $|x'| < \delta$ and $f(x') = (1/x') \geq |L| + 2$. But then $|f(x') - L| \geq 2 > 1$, contradicting the fact that $|f(x) - L| < 1$. Since we have achieved a contradiction, we conclude that $\lim_{x\to 0} f(x)$ does not exist.

We now show: for all $N > 0$, there exists $\delta(N) > 0$ such that: $0 < x < \delta(N)$ implies $f(x) > N$. Let $N > 0$, and define $\delta(N) = 1/N$. Let $0 < x < \delta(N) = 1/N$. Then $1/x > N$, so $f(x) = 1/x > N$, as desired. From (1), $\lim_{x\to 0} f(x)$ does not exist. However, we have shown that f becomes arbitrarily large as $x \to 0^+$. We express this behavior of f with the following notation.

$$\lim_{x\to 0^+} f(x) = \infty.$$

Also, for all $N < 0$, there exists $\delta(N) < 0$ such that: $\delta(N) < x < 0$ implies $f(x) < N$. Let $N < 0$, and define $\delta(N) = 1/N$. Let $1/N = \delta(N) < x < 0$. Then $1/x < N$, so $f(x) = 1/x < N$, as desired. We express this behavior of f with the following notation.

$$\lim_{x\to 0^-} f(x) = -\infty.$$

□

Example 4.4. (**Infinite oscillation**) Define $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Claim: $\lim_{x\to 0} f(x)$ DNE.

Proof of Claim. We will show that $\lim_{x\to 0} f(x)$ does not exist. We argue by contradiction. Suppose $L \in \mathbb{R}$ and $\lim_{x\to 0} f(x) = L$. We now apply the definition (1). Let $\varepsilon = 1/2$. Then there exists $\delta > 0$ such that: $0 < |x| < \delta$ implies $|f(x) - L| < 1/2$. Let n be a positive integer such that $0 < 1/n < \delta$. Let $x_1 = 1/(2\pi n)$ and let $x_2 = 1/(\pi + 2\pi n)$. Since $0 < 1/(2\pi) < 1$, we see that $0 < 1/(\pi + 2\pi n) < 1/(2\pi n) < 1/n < \delta$. That is, $0 < |x_1| < \delta$ and $0 < |x_2| < \delta$. Applying (1), we conclude that

$$|f(x_1) - L| < 1/2, \quad \text{and} \quad |f(x_2) - L| < 1/2.$$

Now, by the definition of f, we have $f(x_1) = \cos(2\pi n) = 1$, and $f(x_2) = \cos(\pi + 2\pi n) = -1$. So, L satisfies $|1 - L| < 1/2$ and $|-1 - L| < 1/2$. That is, L satisfies

$$1/2 < L < 3/2, \quad \text{and} \quad -3/2 < L < -1/2.$$

Since no such L exists, we have achieved a contradiction. We conclude that $\lim_{x\to 0} f(x)$ does not exist. $\square$

5. Calculating Limits

In order to manipulate limits, we need to be able to apply some simple operations to them. The following statements summarize some ways that we can manipulate limits.

Operations on Limits:

Let $L, M, a \in \mathbb{R}$. Assume that $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ exists. Let $L = \lim_{x\to a} f(x)$ and let $M = \lim_{x\to a} g(x)$. Then

(a) $\lim_{x\to a}(f(x) + g(x)) = (\lim_{x\to a} f(x)) + (\lim_{x\to b} g(x)) = L + M$.
(b) $\lim_{x\to a}(f(x) - g(x)) = (\lim_{x\to a} f(x)) - (\lim_{x\to b} g(x)) = L - M$.
(c) $\lim_{x\to a}[(f(x))(g(x))] = (\lim_{x\to a} f(x))(\lim_{x\to b} g(x)) = LM$.
(d) If $M \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to b} g(x)} = \frac{L}{M}$.
(e) If $\alpha \in \mathbb{R}$ and $L > 0$, then $\lim_{x\to a}((f(x))^\alpha) = L^\alpha$

To get an idea for why these things are true, we will prove property (a). Intuitively, if we take x to be close to a, then $f(x)$ will be close to L. Then, if we take x even closer to a if necessary, $g(x)$ will be close to M. So, for x very close to a, $f(x) + g(x)$ is close to $L + M$. This intuition can be turned into a proof.

Exercise 5.1. Try to prove property (c) by adapting the proof below.

Proof of (a). Let $L, M, a \in \mathbb{R}$. Assume that $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ exists. Let $L = \lim_{x\to a} f(x)$ and let $M = \lim_{x\to a} g(x)$. We consider the function $f(x) + g(x)$. Let $\varepsilon > 0$. We need to find $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) + g(x) - L - M| < \varepsilon$. Apply the definition (1) to f with $\varepsilon_1 = \varepsilon/2$ to find $0 < \delta_1$ such that:

$$0 < |x - a| < \delta_1 \quad \text{implies} \quad |f(x) - L| < \varepsilon/2.$$

Apply again the definition (1) to g with $\varepsilon_2 = \varepsilon/2$ to find $0 < \delta_2$ such that:

$$0 < |x - a| < \delta_2 \quad \text{implies} \quad |g(x) - M| < \varepsilon/2.$$

We choose δ to be the minimum of δ_1 and δ_2. Then $0 < |x - a| < \delta$ implies $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$. So, $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon/2$ and $|g(x) - M| < \varepsilon/2$. Applying the triangle inequality,

$$\begin{aligned} |f(x) + g(x) - L - M| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

So, the δ we have chosen satisfies the desired condition of the definition (1) for the function $f(x) + g(x)$. We conclude that $L + M = \lim_{x\to a}(f(x) + g(x))$, as desired. $\square$

6. Continuity

Definition 6.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ let $a \in \mathbb{R}$. We say that f is **continuous at** a if the following three conditions are satisfied

(i) $\lim_{x\to a} f(x)$ exists
(ii) a is in the domain of f
(iii) $\lim_{x\to a} f(x) = f(a)$

We say that a function f is **continuous** if f is continuous on all points of its domain.

Intuitively, f is continuous at a if f does not "jump around too much" around a. In order to better understand continuity, we will give two more equivalent definitions of continuity. The first one is really just an application of the definition (1). Fix $a \in \mathbb{R}$ and let x be a variable. The function f is continuous at a if and only if the following condition is satisfied

$$\begin{gathered}\text{For all } \varepsilon > 0, \text{ there exists } \delta(\varepsilon) > 0 \text{ such that:}\\ \text{if } |x-a| < \delta(\varepsilon), \text{ then } |f(x) - f(a)| < \varepsilon\end{gathered} \tag{2}$$

Unlike in (1), note that (2) allows for x such that $|x - a| = 0$, i.e. we can take $x = a$.

Our third equivalent way of defining continuity may appear a bit strange. However, we will need this formulation to prove the Intermediate Value Theorem.

Definition 6.2. Let $a, b, c, d \in \mathbb{R}$ with $c < d$, $a < b$, and let $f\colon (a, b) \to \mathbb{R}$. Then the set $f^{-1}(c, d) \subseteq \mathbb{R}$ is defined by

$$f^{-1}(c,d) = \{x \in \mathbb{R}\colon f(x) \in (c,d)\}$$

Lemma 6.3. *Let $a, b \in \mathbb{R}$, and let $f\colon (a, b) \to \mathbb{R}$. The following conditions are equivalent*

(i) *f is continuous.*
(ii) *For any open interval (c, d) with $c, d \in \mathbb{R}$, $c < d$, the set $f^{-1}(c, d)$ is a union of open intervals.*

Proof. We first prove that (i) implies (ii). Assume that f is continuous. Let $c, d \in \mathbb{R}$ with $c < d$. Fix $x \in f^{-1}(c, d)$. By Definition 6.2, $f(x) \in (c, d)$. To prove (ii), it suffices to show that x is contained in some interval $(x - A, x + A)$ such that $(x - A, x + A) \subseteq f^{-1}(c, d)$, and $(x - A, x + A) \subseteq (a, b)$. Let ε be the minimum of $|f(x) - c|/2$ and $|f(x) - d|/2$. Applying the definition (2) to f at x, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $A > 0$ be the minimum of $\delta/2$, $|x - a|/2$ and $|x - b|/2$. Then $(x - A, x + A) \subseteq (a, b)$. Also, since $0 < A < \delta/2$, if $z \in (x - A, x + A)$, then $|f(x) - f(z)| < \varepsilon$. By our choice of ε, the previous inequality implies that $(x - A, x + A) \subseteq f^{-1}(c, d)$. We conclude that (ii) holds.

We now prove that (ii) implies (i). Let $\varepsilon > 0$ and let $x \in (a, b)$. We need to find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. From (ii), the set $f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon)$ is a union of open intervals. So, since $x \in f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon)$, we can find $c, d \in \mathbb{R}$ with $c < d$ such that $x \in (c, d)$ and $(c, d) \subseteq f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon)$. Since $x \in (c, d)$, we can find $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (c, d)$. Then

$$(x-\delta, x+\delta) \subseteq (c,d) \subseteq f^{-1}(f(x)-\varepsilon, f(x)+\varepsilon).$$

That is, $|x - y| < \delta$ implies $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$. That is, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Since we have found our desired δ, we conclude that f is continuous, and the proof of (i) is complete.

Since (i) implies (ii), and (ii) implies (i), we conclude that (i) and (ii) are equivalent. □

7. A Proof of the Intermediate Value Theorem

The Intermediate Value Theorem is the first "real" theorem that we have seen in this course. Its statement may have some intuition behind it, but it may not be immediately obvious how to prove it. For this reason, and to deepen our understanding of the subject matter, we present a proof of the theorem. At the heart of the argument is the following Lemma, which asserts that, in some sense, open intervals are connected. The proof of the intermediate value theorem then proceeds by showing that a continuous function must map a connected set to another connected set.

Lemma 7.1. *Let $a,b\in\mathbb{R}$, $a<b$. Then the open interval (a,b) cannot satisfy $(a,b)=C\cup D$, where $C\cap D=\emptyset$, $C\neq\emptyset$, $D\neq\emptyset$, and C,D are each unions of open intervals.*

Proof. We argue by contradiction. Let C,D be unions of open intervals such that $C\cap D=\emptyset$, $C\neq\emptyset$, $D\neq\emptyset$, and $(a,b)=C\cup D$. Let $c\in C$ and $d\in D$. Without loss of generality, $c<d$. Since C is a union of open intervals, there exists $\varepsilon>0$ such that $(c-\varepsilon,c+\varepsilon)\subseteq C$. Let c' be the least upper bound of the set $\{z\in\mathbb{R}\colon [c,z)\subseteq C\}$. Note that $c'\leq b$. Since there exists $\varepsilon>0$ such that $(c-\varepsilon,c+\varepsilon)\subseteq C$, we conclude that $c'>c$.

We now split into two cases. In each case, we will achieve our desired contradiction.

Case 1. $c'<b$. Since C is a union of open intervals, $c'\notin C$. (If $c'\in C$, then there exists $\varepsilon'>0$ such that $(c'-\varepsilon',c'+\varepsilon')\subseteq C$. This follows since C is a union of open intervals. But this containment contradicts the definition of c'.) Since $c'\notin C$ and $(a,b)=C\cup D$ with $c'<b$ and $a<c<c'$, we conclude that $c'\in D$. But since D is a union of open intervals, there exists $\varepsilon'>0$ such that $(c-\varepsilon',c+\varepsilon')\subseteq D$. But this containment contradicts the definition of c'.

Case 2. $c'=b$. Since we assumed that $c<d$ and $d\in D\subseteq(a,b)$, we have $c<d<b$, but $b=c'$, so $c<d<c'$. But this inequality contradicts the definition of c'.

In any case, we have found a contradiction. We conclude that no such C,D exist, and therefore the Lemma holds, as desired. □

Theorem 7.2. ***(Intermediate Value Theorem)*** *Let $a,b\in\mathbb{R}$, $a<b$. Let $f\colon[a,b]\to\mathbb{R}$ be continuous. Then f achieves every value between $f(a)$ and $f(b)$. That is, for every $y\in\mathbb{R}$ in between $f(a)$ and $f(b)$, there exists $x\in[a,b]$ such that $f(x)=y$.*

Proof. We argue by contradiction. First, without loss of generality, assume that $f(a)\leq f(b)$. (If this is not the case, replace f with $-f$.) Now, assume for the sake of contradiction that $y\in[f(a),f(b)]$, but there does not exist $x\in[a,b]$ such that $f(x)=y$. Since $f(a)\in[f(a),f(b)]$ and $f(b)\in[f(a),f(b)]$, we may assume that $y\in(f(a),f(b))$.

Now, since $(f(a),y)$ and $(y,f(b))$ are disjoint sets, $f^{-1}(f(a),y)$ and $f^{-1}(y,f(b))$ are disjoint sets. Since f is continuous at a, $(a,b)\cap[f^{-1}(f(a),y)]\neq\emptyset$. Since f is continuous at b, $(a,b)\cap[f^{-1}(y,f(b))]\neq\emptyset$. Since $f\colon(a,b)\to\mathbb{R}$ is continuous, Lemma 6.3 says that $f^{-1}(f(a),y)$ and $f^{-1}(y,f(b))$ are each unions of open intervals. Since there does not exist $x\in[a,b]$ such that $f(x)=y$, we conclude that

$$(a,b)=\left(f^{-1}(f(a),y)\right)\cup\left(f^{-1}(y,f(b))\right)$$

But this equality contradicts Lemma 7.1. Since we have achieved a contradiction, we conclude that some x exists such that $f(x)=y$, as desired. □

8. Selected Exercises

The following exercises from the textbook are intended to sharpen your knowledge of the details of the above concepts.

Exercise 8.1. Evaluate the following limit and justify each step by indicating the appropriate limit law.

$$\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$$

Exercise 8.2. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

$$\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$$

Exercise 8.3. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

$$\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

Exercise 8.4. Is there a real number a such that the following limit exists?

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

If so, find the value of a and the value of the limit.

9. Selected Problems

The following problems are intended to deepen your understanding of the concepts discussed above. In the future we will try to include problems that are based upon applications of Calculus.

Problem 9.1. Are the following statements true or false?

(a) If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} \frac{f(x)}{g(x)}$ does not exist.
(b) If x is a real number, then $\sqrt{x^2} = x$
(c) If $\lim_{x \to 5} f(x) = 2$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} \frac{f(x)}{g(x)}$ does not exist.
(d) If f is continuous at 5 and $f(5) = 2$, then $\lim_{x \to 2} f(4x^2 - 11) = 2$.
(e) If $f(x) > 1$ for all $x \neq 0$ and $\lim_{x \to 0} f(x)$ exists, then $\lim_{x \to 0} f(x) > 1$.

Problem 9.2. Fix $x \in \mathbb{R}$, and let $f(x) = x^2$. Calculate the following limit

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

The fraction $(f(x+h) - f(x))/h$ is known as a difference quotient. The limit of this difference quotient will come up again in Chapter 2.

Problem 9.3. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. Is it always true that $\lim_{x \to a}(f(x) + g(x)) = (\lim_{x \to a} f(x)) + (\lim_{x \to a} g(x))$?

Problem 9.4. Does there exist a function $f \colon \mathbb{R} \to \mathbb{R}$ that is discontinuous everywhere? In other words, is there a function $f \colon \mathbb{R} \to \mathbb{R}$ such that: for every $x \in \mathbb{R}$, f is not continuous at x?

Problem 9.5. Show that the function $f(x) = x^3 - x - 1$ has a zero between -1 and 2.

Problem 9.6. Let $x, y \in \mathbb{R}$. Draw the following set and describe it in words:

$$\{(x,y) \in \mathbb{R}^2 \colon \lim_{t\to\infty} (|x|^t + |y|^t) < 4\}.$$

10. Appendix: Notation

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of integers

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$, the set of natural numbers

$\mathbb{Q} = \{\ldots, 1/2, -2/3, 0, \ldots\} = \{m/n \colon m, n \in \mathbb{Z}, n \neq 0\}$, the set of rational numbers

$\mathbb{R} = \{\ldots, 1/2, \pi, \sqrt{3}, -.24534, -1, 4, 7.2, \ldots\}$, the set of real numbers

$\emptyset$ denotes the empty set, the set consisting of zero elements

$\in$ means "is an element of." For example, $2 \in \mathbb{Z}$ is read as "2 is an element of $\mathbb{Z}$."

$\forall$ means "for all"

$\exists$ means "there exists"

Remark 10.1. In the book, there are several expressions of the form

$$f(x) = \frac{\cos 2x - x}{x^2}$$

When the cosine is written without parentheses in the argument, it is usually understood that the very first thing that is written after the cosine (in this case $2x$) is the argument of the cosine. The next minus or plus sign that appears is assumed to occur outside of the parentheses (that have been omitted). That is, the expression above can be equivalently written as follows

$$f(x) = \frac{\cos(2x) - x}{x^2}$$

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