Please provide complete and well-written solutions to the following exercises.

Due March 6, at the beginning of class.

Assignment 8

Exercise 1. Let $F(x,y) = (x^2 + 4y, x + y^2)$ be a vector field in the plane, and let C be the square in the plane bounded by the lines $x = 0, x = 1$ and by the lines $y = 0, y = 1$. Using Green's Theorem, compute the quantity $\oint_C F \cdot T\, ds$, where the integral is oriented counterclockwise.

Exercise 2. Let $F(x,y) = (y^2 - x^2, x^2 + y^2)$ be a vector field in the plane, and let C be triangle in the plane bounded by the lines $y = 0$, $x = 3$ and $y = x$. Using Green's Theorem, compute the quantity $\oint_C F \cdot T\, ds$, where the integral is oriented counterclockwise.

Exercise 3. Let C be a simple closed curve in the plane. Let D be a region in the plane so that D has boundary C. By choosing the vector field $F(x,y) = (1/2)(-y, x)$, using Green's Theorem, and noting that $\mathrm{curl} F(x,y) = 1$, note that the area of D is equal to the following line integral, which is oriented counterclockwise.

$$(1/2)\oint_C (-y, x) \cdot T\, ds.$$

Using this fact, compute the area of the ellipse $s(t) = (a\cos t, b\sin t)$, where $0 \leq t \leq 2\pi$, and a, b are positive constants.

Exercise 4. Let C denote a square in the plane, oriented counterclockwise. Show that the value of

$$\oint_C (xy^2, x^2y + 2x) \cdot T\, ds$$

around any square depends only on the area of the square, and not on the location of the square in the plane.

Exercise 5. Let C denote five connected straight line segments which traverse the following vertices in order: $(-1, 0)$, then $(2, 0)$, then $(2, 1)$, then $(-5, 1)$, then $(-5, -1)$, then $(-1, -1)$. Compute the line integral of $F(x,y) = (x^3, 4x)$ in the plane, counterclockwise along C. To save time, use Green's Theorem to relate this line integral to the line integral along the straight path from $(-1, -1)$ to $(-1, 0)$.

Exercise 6. Let C denote the lemniscate curve defined by $(x^2 + y^2)^2 = xy$, where $x, y \geq 0$. Find a parametrization of C using the parameter $\theta = \tan^{-1}(y/x)$. Then, using the area formula from Exercise 3, compute the area enclosed by C. (Hint: consider using $x = \cos^{3/2}\theta \sin^{1/2}\theta$)

Exercise 7. Let D be an annulus in the plane. Specifically, assume that, the boundary of D consists of an outer circle C_1 oriented counterclockwise of radius 5, and an inner circle C_2 oriented clockwise of radius 2. Assume that these circles are centered at the origin. Let

$F\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a vector field such that $\mathrm{curl}(F)(x,y) = -3$. Assume that $\oint_{C_2} F \cdot T\, ds = 12$. Using Green's Theorem, find $\int_{C_1} F \cdot T\, ds$.

Exercise 8. Let C be the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, where we orient the triangle counterclockwise as viewed from above. Let $F(x,y,z) = (y^2 + z^2, x^2 + z^2, x^2 + y^2)$. Using Stokes' Theorem, calculate $\oint_C F \cdot T\, ds$.

Exercise 9. Let S be a surface parametrized by $G(r,\theta) = (r\cos\theta, r\sin\theta, r)$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. We denote e_n as the vector pointing in the same direction as $\partial G/\partial r \times \partial G/\partial\theta$ but with unit length. Let $F(x,y,z) = (x^2y, y^3z, 3z)$. Using Stokes' Theorem, calculate $\iint_S \mathrm{curl}(F) \cdot e_n\, dS$.

Exercise 10. Let $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ be a vector field. Let S be a sphere. Let e_n denote the exterior unit normal of S. Compute:

$$\iint_S (\mathrm{curl}(F)) \cdot e_n\, dS.$$