Please provide complete and well-written solutions to the following exercises.

Due February 20, at the beginning of class.

Assignment 7

Exercise 1. Integrate the function $f(x,y,z) = yz$ over the part of the sphere $x^2+y^2+z^2 = 4$ that lies above the cone $z = \sqrt{x^2+y^2}$.

Exercise 2. Let $f(x,y,z) = x^2$. Compute $\iint_S f\,dS$ where S is the surface $x^2+y^2+z^2 = 1$ with $x,y,z \geq 0$.

Exercise 3. Find the surface area of the part of the cone $z^2 = x^2+y^2$ between the planes $z = 1$ and $z = 4$.

Exercise 4. Let a,b,c be positive constants. An ice cream cone is defined as the surface $z = a\sqrt{x^2+y^2}$ where $z \leq b$. Suppose the ice cream cone has surface area c. Find the ice cream cone of fixed surface area c and with maximum volume. (This way, you get to eat the most ice cream with the least amount of material.)

Exercise 5. Let (x,y,z) be a point in Euclidean space $\mathbf{R}^3$. Let G,m be constants. Let S denote the sphere of radius R centered at the origin. Let dS denote the surface area element of S with respect to variables a,b,c. Define the following function, which is the gravitational potential of S at the point (x,y,z).

$$V(x,y,z) = -\frac{Gm}{4\pi R^2}\iint_S \frac{dS}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} = -\frac{Gm}{4\pi R^2}\iint_S \frac{dS}{||(x,y,z)-(a,b,c)||}.$$

- Using a symmetry argument, show that $V(x,y,z)$ only depends on $||(x,y,z)||$. That is, if $\Phi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a rotation, then $V(x,y,z) = V(\Phi(x,y,z))$. (Hint: it may be helpful to write $(a,b,c) = \Phi(\Phi^{-1}(a,b,c))$, and then to use $||\Phi(d,e,f)|| = ||(d,e,f)||$, and also use $\Phi((a,b,c)+(d,e,f)) = \Phi(a,b,c)+\Phi(d,e,f)$. Then, use that dS does not change when we apply the rotation Φ^{-1}. That is, if $f(a,b,c)$ is a function, you may assume that $\iint_S f(a,b,c)dS = \iint_S f(\Phi^{-1}(a,b,c))dS$.) So, to compute V at any point, it suffices to compute $V(0,0,r)$ for any $r \geq 0$. That is, we have shown that $V(x,y,z) = V(0,0,||(x,y,z)||)$.
- Let $r \geq 0$. Using spherical coordinates, show that
$$V(0,0,r) = -\frac{Gm}{4\pi}\int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} \frac{\sin\phi\,d\theta d\phi}{\sqrt{R^2+r^2-2Rr\cos\phi}}.$$
- Using the substitution $u = R^2+r^2-2Rr\cos\phi$, show that
$$V(0,0,r) = -\frac{mG}{2Rr}(|R+r|-|R-r|).$$

- Verify that V satisfies the following formula

$$V(x,y,z) = \begin{cases} -Gm/\,||(x,y,z)|| & \text{, if } ||(x,y,z)|| > R \\ -Gm/R & \text{, if } ||(x,y,z)|| < R \end{cases}.$$

  In particular, a hollow sphere exerts no gravitational force inside the sphere.

Exercise 6. Let $F(x,y,z) = (x,y,z)$ be a vector field. Let a be a real number. Compute the flux $\iint_S F \cdot e_n \, dS$ outward through the surface S where $x^2+y^2=1$ and $0 \leq z \leq a$.

Exercise 7. Find the flux of the vector field $F(x,y,z) = (xze^y, -xze^y, z)$ through the part of the plane $x+y+z=1$ that lies in the first octant, where the flux is oriented downwards. (That is, you should choose the normal vector with negative z component.)

Exercise 8. Let $F(x,y,z) = (x,y,e^z)$ be a vector field. Compute $\iint_S F \cdot e_n \, dS$ where S is the surface $x^2+y^2=9$, $1 \leq z \leq 4$, and e_n denotes the outward pointing unit normal vector to S.