Please provide complete and well-written solutions to the following exercises.

Due January 30, at the beginning of class.

Assignment 4

Exercise 1. Let $G(u, v) = (3u+v, u-2v)$. Using the Jacobian, determine the area of $G(D)$ where D is defined as

- $D = [0, 3] \times [0, 5]$.
- $D = [2, 5] \times [1, 7]$.

Exercise 2. Evaluate the integral

$$\int_{z=0}^{z=3} \int_{y=0}^{y=4} \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dxdydz$$

by the applying the transformation $u(x, y, z) = (2x-y)/2$, $v(x, y, z) = y/2$, and $w(x, y, z) = z/3$.

Exercise 3. Let $d, e, f > 0$ be constants. Consider the ellipsoid, defined as the set of (a, b, c) in Euclidean space $\mathbf{R}^3$ such that $a^2/d^2 + b^2/e^2 + c^2/f^2 \leq 1$. Compute the volume of this ellipsoid in terms of d, e, f. (Hint: let D denote the ball $x^2 + y^2 + z^2 \leq 1$, then define $g(x, y, z) = (dx, ey, fz) = (a(x, y, z), b(x, y, z), c(x, y, z))$.)

Exercise 4. Find the center of mass of the region in the plane that is bounded by the positive x axis, the positive y axis, and the graph of the function $y = e^{-x}$. (Assume the region has constant density 1.)

Exercise 5. Let D be the region in the plane lying between the curve $x = y^2 + 1$ and the line $x = 3$. Compute the average squared distance from the origin $(0, 0)$ to the region D.

Exercise 6. This exercise shows that multivariable calculus can prove facts about single variable functions. Let f be a function of the variable y. Assume that f is continuous. For any real t, define $G(t) = \int_{x=0}^{x=t} \int_{y=0}^{y=x} f(y)dydx$.

- Using the Fundamental Theorem of Calculus, show that $G''(t) = f(t)$.
- By changing the order of integration, show that $G(t) = \int_0^t (t-y)f(y)dy$. That is, the second antiderivative of f can be expressed as an integral involving f.

Exercise 7. Find the Jacobian determinant of the following map.

$$G(v, w) = (v^2 + w^2, v^2 - w^2).$$

Exercise 8. Let D be the region in the plane where $x > 0$, $y > 0$, and which is bounded by the curves $xy = 1$, $xy = 9$, $y = x$ and $y = 4x$. Using the transformation $x = u/v$ and $y = uv$

with $u > 0$ and $v > 0$, evaluate the following integral.

$$\iint_D \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dxdy.$$

(Hint: use $G(x, y) = (\sqrt{xy}, \sqrt{y/x}) = (u(x, y), v(x, y))$.)

Exercise 9. Let D denote the plane $z = 0$ in Euclidean space $\mathbf{R}^3$. Let S denote the unit sphere $x^2 + y^2 + z^2 = 1$. Consider the function $G: D \to S$ where $G(x, y, 0)$ is defined to be the point of intersection of S with the straight line that goes through $(x, y, 0)$ and $(0, 0, 1)$.

- Find a formula for G in terms of x and y. That is, find functions $u(x, y), v(x, y)$ and $w(x, y)$ such that $G(x, y, 0) = (u(x, y), v(x, y), w(x, y))$. (Hint: First draw a picture. Let L denote the line that goes through $(x, y, 0)$ and $(0, 0, 1)$. We want to find the intersection of L and the unit sphere S. Consider the triangle formed by $(0, 0, 0)$, $(0, 0, 1)$ and $(x, y, 0)$. This right triangle has base $\sqrt{x^2 + y^2}$, height 1 and hypotenuse $\sqrt{1 + x^2 + y^2}$. Consider the intersection of L with the sphere S. Suppose L intersects the point $w = (a, b, c)$ in S. Consider the triangle formed by $(0, 0, 1)$, $(0, 0, c)$ and w. This right triangle is similar to the first right triangle. Also, the new right triangle has base $\sqrt{a^2 + b^2}$, height $1 - c$ and hypotenuse $\sqrt{a^2 + b^2 + (1 - c)^2}$. We want to find (a, b, c) in terms of x and y. To do this, first use the similarity of the right triangles to get $x(1 - c) = a$ and $y(1 - c) = b$. Now, you should be able to solve for c in terms of x and y.)
- Is G a one-to-one correspondence?

The function G^{-1} is a projection of the sphere onto the plane. There are many ways to map the sphere onto flat surfaces, such as the Mercator projection. As we know intuitively from our understanding of maps of the earth, it is impossible to map the sphere onto the flat plane without distorting some of the distances on the sphere. (This can be proven, but we cannot do so in this class. Such a result deals with the curvature of the sphere and of the plane.)