Digest 8

(A compilation of emailed homework questions, answered around Wednesday.) (Sending this out Monday, may update in one or two days.)

Question. (From a student): What is covered on the second exam for this class?

Answer. See the CCLE announcement labelled "Exam 2 Stuff."

Question. (From a student): Whenever you integrate an ice cream cone, do you always parametrize with spherical coordinates with ρ from 0 to $\pi/4$, θ from 0 to 2π because even though you have a cone in there, it's basically like you're slicing out 1/4 of the sphere?

Answer. The short answer is no. First of all, I assume you mean that we are trying to find the volume enclosed by the cone $z = a\sqrt{x^2 + y^2}$ where a, b are constants and $z \leq b$. In this case, if you want to find the volume using spherical coordinates, recall that $\phi = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$. So, if we are on the surface of the cone, we have $x^2 + y^2 = z^2/a^2$, so $\phi = \cos^{-1}(z/\sqrt{z^2(1 + 1/a^2)}) = \cos^{-1}(1/\sqrt{1 + 1/a^2}) = \cos^{-1}(a/\sqrt{a^2 + 1})$. (Note that we used $z \geq 0$.) So, in the case that $a = 1$, we have $\phi = \cos^{-1}(1/\sqrt{2}) = \pi/4$. However, for different values of a, $\phi \neq \pi/4$. So, if you want to evaluate the volume using spherical coordinates, your limits on ϕ would generally not have to be from 0 to $\pi/4$. Also, note that in this case, the volume is probably easier to compute using cylindrical coordinates (see the homework solutions for details).

Question. If we can't calculate 3D flux integrals, how do we calculate flux of 3D things in real life?

Answer. I'm not sure I understand this question. We have discussed how we can compute the flux of a vector field $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ through a two-dimensional surface S. What does it mean to compute the flux of a vector field $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ through a three-dimensional object? This question seems to have no answer. (If you do not see why, just think about it for a while, or try yourself to make such a definition.) A simpler question has a similar problem. What does it mean to compute the flux of a vector field $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ through a domain D in the plane? Once again, this question seems to have no answer.

The definition of flux requires that we define a unit normal vector. For a two-dimensional surface S sitting in $\mathbf{R}^3$, we can define (exactly two) unit normal vectors to S. For a curve C sitting in $\mathbf{R}^2$, we can define (exactly two) unit normal vectors to C. In both cases, we can define a notion of flux. In general, if you want to define flux, you need to have some definition of a unit normal vector, which usually requires that your surface (or curve) has one less dimension than the ambient space.

It is possible to define flux of a vector field $F\colon \mathbf{R}^4 \to \mathbf{R}^4$ through a three-dimensional "surface" S in $\mathbf{R}^4$, but this is beyond what we will cover in this class, and I will not make this statement precise.

Question. (From a student): I was wondering if you could explain these topics through more conceptually and maybe provide some examples.

- line integrals
- surface integrals
- flux
- some simple steps to parametrize a path

Answer. Any integral is essentially a sum of a bunch of values of a function. Since an integral is a limit of Riemann sums, and a Riemann sum is a sum of a bunch of values of a function, we can therefore intuitively think of an integral as a sum of a bunch of values of a function. A line integral is an integral of a function f on a line. So, we can think of a line integral as the sum of a bunch of values of a function on that line. For example, $\int_C f ds$ integrates a function f on a curve C, and $\int_C F \cdot T ds$ integrates the function $F \cdot T$ on a line, where F is a vector field and T is a unit tangent vector to a curve C. Similarly, a surface integral is an integral of a function f on a surface. So, we can think of a surface integral as the sum of a bunch of values of a function on that surface. For example, $\iint_S f dS$ integrates a function f on a surface S, and $\iint_S F \cdot e_n dS$ integrates the function $F \cdot e_n$ on the surface S, where F is a vector field and e_n is a unit normal vector to S. The integral $\iint_S F \cdot e_n dS$ is also called a flux integral of a surface. We can also interpret this integral as the amount of flow of the vector field across S in the direction e_n. For more specific examples, see the Examples in the lecture notes here.

I'm not sure if there is a single method to parameterize a path. You should be able to do standard parameterizations, e.g. parameterizing a line, a circle, an ellipse (going in both directions in each case). Most examples will be variations of these, as we have seen on the homework, or a parametrization will be given in the problem. For example, if you have looked at Homework 8, Exercise 6, the parametrization is kind of tricky, so it is given as part of the problem.

Question. How do we know when to use Green's and when to just compute the line integral?

Can you give some examples of complicated surface integral parametrizations? (I had trouble with some of the midterm questions)

Answer. For the first question, I'm not sure there is a general method, and sometimes you just have to try both. Sometimes the boundary curve might be really complicated, or it might involve several different pieces (e.g. a square). In that case, it might be better to use Green's Theorem, since integrating on the inside of a square is easier than integrating four separate line integrals on the boundary of the square. So there could be different things going on. If the line integral looks easy enough to do, then using Green's Theorem would not be necessary.

For the second question, first just note that these other practice exams often have different (i.e. longer) time constraints, so they can ask questions that will take longer in total then our shorter fifty minute exam.

Now, let's first go through the things we did in class: a few different parametrizations of the sphere; a general parametrization for a surface of the form $z = f(x, y)$; a few parametrizations of the cone and double cone; the cylinder; the parabolic cylinder. In the practice exams, you may have run into the following examples.

(1) The bottom sheet of the two-sheeted hyperboloid $x^2 + y^2 = z^2 - 1$. This example falls into the category of a surface defined by $z = f(x, y)$ (if you solve for z in the right way), so we know how to do this one. That is, we could solve for z to get $z = \pm\sqrt{x^2 + y^2 + 1}$, so we use the parameterization $G(x, y) = (x, y, -\sqrt{x^2 + y^2 + 1})$. (We chose the minus sign on the z-component since we want to parametrize the bottom part of the paraboloid.) Alternatively, if you notice that you are summing two squares, you can use a variation on polar coordinates with the parametrization $G(r, \theta) = (r\cos\theta, r\sin\theta, -\sqrt{1 + r^2})$. Note that $(r\cos\theta)^2 + (r\sin\theta)^2 = (-\sqrt{1 + r^2})^2 - 1$.

(2) The curve $z = \sin y$ rotated about the y-axis. Rotating about the y-axis means that the distance from the y-axis is fixed, and if the point has fixed y-coordinate, then we are rotating around in the zx-plane. So, if we have a point $(0, y, \sin(y))$ in the curve, then the point (x, y, z) is also in the curve, where (x, y, z) is supposed to have distance $\sin(y)$ from the y-axis. That is, we need $x^2 + z^2 = \sin^2(y)$. So, we can parametrize all such points by $(x, y, z) = ((\sin y)\cos\theta, y, (\sin y)\sin\theta)$ with $0 \leq \theta < 2\pi$, since then $x^2 + z^2 = \sin^2(y)$, as desired. That is, we can use the parametrization $G(y, \theta) = ((\sin y)\cos\theta, y, (\sin y)\sin\theta)$.

Anyway, there are kind of an infinite number of examples, so maybe I could just give some general advice. Probably the first thing you want to do is you might have some equation for your surface, and e.g. you can try to solve for one variable (maybe z), so that you can write $z = f(x, y)$ and then use the parametrization $G(x, y) = (x, y, f(x, y))$. This is basically one of the few general procedures you can do, but it will work for almost all problems. If it does not work, maybe you could try to split up your surface into different pieces over which this method works. For example, the sphere $x^2 + y^2 + z^2 = 1$ can be parametrized by using one parametrization for the top of the sphere $G(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ and one for the bottom of the sphere $H(x, y) = (x, y, -\sqrt{1 - x^2 - y^2})$.

Also, sometimes things get easier maybe if you use the trick that I mentioned above about the sum of two squares, i.e. if there is some rotational symmetry (as in example (2)), then this should clue you in that maybe you should use sin and cos somewhere. Sometimes this isn't necessary, but it just makes things simpler. For example, you can use a single parametrization to parametrize the unit sphere as $G(z, \theta) = (\cos\theta\sqrt{1 - z^2}, \sin\theta\sqrt{1 - z^2}, z)$ where $0 \leq \theta < 2\pi$ and $-1 \leq z \leq 1$. These are some of the only tricks that I know, but they are probably enough to cover any examples you come across.

Question. I know if you reverse direction with vector field line integrals, it matters. The parametrization you do matters for vector fields as well. But does this also matter for flux integrals because you're integrating over a vector field in that too? Also how can you every get a definitive answer if changing your parametrizations gives you a different answer every time?

Also in HW#6 problem #6, why do you have to use two line integrals? The vector field is conservative why can't you just use the two endpoints?

Answer. Reversing the direction of $\int_C F \cdot T ds$ changes the sign of the integral, basically because T reverses its direction. On the other hand, $\int_C F \cdot e_n ds$ does not change sign when you reverse the direction, as long as e_n is still pointing in the same direction. That is what occurs for reversing the direction. However, except for the direction in the case of $\int_C F \cdot T$, the details of the parametrization *do not change the outcome of the integral.* This was the point of Homework 5, Exercise 1. I would encourage you to look closely at exactly what this Exercise says, since it should answer your question. The same goes for flux integrals on surfaces. The parametrization you choose for the surface does not change the answer, as long as the normal vector points in the desired direction.

For homework 6 exercise 6, yes, I agree you can use the endpoints; this was what was done in part (b). Part (a) was just meant to emphasize that part (b) is the simpler way of doing things.

Question. To figure out the normal vector to a line in order to compute the flux integral, do I just think about what dot product $n(t).s`(t) = 0$? In other words, is there no algebraic way to solve for the normal vector? What if it is hard to find a vector in which the dot product will be 0?

Answer. Suppose $s\colon [a, b] \to \mathbf{R}^2$ parametrizes a curve in the plane. We write $s(t) = (x(t), y(t))$. Using our definition of the flux integral, we have $\int_C F \cdot e_n ds = \int_{t=a}^{t=b} F(s(t)) \cdot n(t) dt$, where $n(t)$ is normal to the curve s, and $n(t)$ has the same length as $s'(t)$. So, we choose either $n(t) = (-y'(t), x'(t))$, or $n(t) = (y'(t), -x'(t))$, according to the desired direction of the normal. So, this is certainly an algebraic formula for n, and it also satisfies $n(t) \cdot s'(t) = 0$.

Question. I noticed sometimes the question is worded "find a potential function" or "find the potential function" for gradients. Can we leave out the C when it says find a potential function because C can be zero and it is a possible function to have no constant at the end? and when it says find the potential function, we would have to put C because that represents every possible potential function?

Answer. This distinction is not important to me. You can take or leave the constant C. In Calc 2, the constant C is emphasized just so everyone remembers that the derivative of any constant function is zero, but since we all remember that, I don't care about the constant C anymore.

Question. Do we need to know proofs of things on the exams?

Answer. It is not necessary to know the proofs, I just think the proofs can help us remember various things, and understand why certain things are true, so this is why I am going through some of them in class. Moreover, some of the ideas that I am presenting will return in later courses, if you take certain other courses.