Digest 7, Homework 7

(A compilation of emailed homework questions, answered around Wednesday.) (Sending this out Tuesday, will probably update Wednesday.)

Question. [Exercise 1] Integrate the function $f(x,y,z)=yz$ over the part of the sphere $x^2+y^2+z^2=4$ that lies above the cone $z=\sqrt{x^2+y^2}$.

(From a student:) If we decide to use spherical coordinates to describe the surface, are we required to input the Jacobian value for Cartesian to Spherical transformations within our integral?

Answer. If you want to use spherical coordinates, then you are using the parametrization $G(\phi,\theta)=(\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi)$, where ϕ and θ lie in an appropriate domain D. We then defined $\iint_S f dS = \iint_D f(G(\phi,\theta))\,||\partial G/\partial\phi\times\partial G/\partial\theta||\,d\phi d\theta$. So, you see there is no Jacobian within this definition, since we are just applying a definition which says nothing about any Jacobian. However, intuitively, the term $||\partial G/\partial\phi\times\partial G/\partial\theta||$ is kind of acting like a Jacobian, and indeed, in this case you should deduce that it is equal to $\sin\phi$, so it is actually agreeing with the Jacobian determinant that we use to move from Cartesian to spherical coordinates (noting that $\rho=1$ in this case).

Finally, in Remark 5.43 in the notes, we can actually deduce the 2-dimensional change of variables formula just using our definition of a surface integral of a function. (Thanks to a student for asking about something along these lines.) So, the surface integral and change of variables formula are quite closely related.

Question. [Exercise 4] Let a,b,c be positive constants. An ice cream cone is defined as the surface $z=a\sqrt{x^2+y^2}$ where $z\leq b$. Suppose the ice cream cone has surface area c. Find the ice cream cone of fixed surface area c and with maximum volume. (This way, you get to eat the most ice cream with the least amount of material.)

(From a student): I don't understand what to do. I found the surface area to be $b^2\pi\times\sqrt{a^2+1}/a^2$, and I set it equal to c. Then, I put it into the volume equation to get $V=bc/3\sqrt{a^2+1}$. Am I supposed to take a derivative and set to zero to maximize the volume with respect to c?

Answer. I don't think substituting c into the volume equation is the correct thing to do. It seems to me that you should get two expressions for the volume and surface area in terms of a and b, and then you apply Lagrange Multipliers. This calculation can be done directly, but I think the calculation is somewhat simplified if you replace the variables a,b with the variables $r=(b/a)$ and b. With these new variables, the cone has volume $(\pi/3)r^2b$ and surface area $\pi r\sqrt{r^2+b^2}$. And then once again, you have a Lagrange Multiplier problem (in the variables r, b); I then recommend using the constraint $\pi^2 r^2(r^2+b^2) = c^2$.

Question. [Exercise 5] Let (x, y, z) be a point in Euclidean space $\mathbf{R}^3$. Let G, m be constants. Let S denote the sphere of radius R centered at the origin. Let dS denote ....

(From a student): Is the first step to find a parametrization Φ?

Is the formula the same for flux integrals on surfaces and vector field integrals on surfaces (whereas for line integrals, they are different)?

Answer. The map Φ is a rotation, not a parametrization. I think I listed all the steps you need to do in the Hint. You should not write a formula for Φ, you should only need to use some symbolic manipulation of Φ.

Concerning your second question, yes, I think you are correct. We only made one definition for integrating a vector field on a surface. And we can interpret this definition as a flux integral. As for line integrals in the plane, we have two different ways to integrate a vector field on the line. We can either integrate $F \cdot T$ or $F \cdot e_n$. However, note that for line integrals in $\mathbf{R}^3$, we only have one way to integrate vector fields, namely by integrating $F \cdot T$.

Question. (From a student:) When a mathematician writes: "Let $f\colon [a, b] \times [c, d] \to \mathbf{R}$ or $[a, b] \to \mathbf{R}$" or even "$[a, b] \to \mathbf{R}^3$" what does he or she mean, in plain English? What does "Let $G\colon D \to S$ be a function" or "Let $f\colon S \to \mathbf{R}$" or "$S = \{G(y)\colon y \in D\}$" even translate to (into words)? What about: "$R^2 \to [0, \infty)$"; "Let $G\colon \mathbf{R}^2 \to \mathbf{R}^2$"; "Let $f\colon \mathbf{R}^3 \to \mathbf{R}$ be a real-valued function on Euclidean space $\mathbf{R}^3$"? Generally, is $\mathbf{R}$ a symbol for real numbers while $\mathbf{R}^2$ and $\mathbf{R}^3$ mean real numbers in 2D and 3D space, respectively?

Answer. The most general statement here is $G\colon D \to S$. This statement is read as "G is a function with domain D and range S." As a specific example, we read $f\colon \mathbf{R} \to \mathbf{R}$ as "f is a function with domain $\mathbf{R}$ and range $\mathbf{R}$." The symbol $\mathbf{R}$ denotes the set of real numbers. The symbol $\mathbf{R}^2$ denotes the plane, and the symbol $\mathbf{R}^3$ denotes three-dimensional Euclidean space. More formally, $\mathbf{R}^2$ is the set of ordered pairs (x, y) where $x \in \mathbf{R}$ and $y \in \mathbf{R}$. And $\mathbf{R}^3$ is the set of ordered pairs (x, y, z) where $x \in \mathbf{R}$, $y \in \mathbf{R}$ and $z \in \mathbf{R}$. So, $f\colon \mathbf{R}^3 \to \mathbf{R}$ means that f is a function with domain $\mathbf{R}^3$ and range $\mathbf{R}$. The statement $S = \{G(y)\colon y \in D\}$ is read as "S is the set of all points of the form $G(y)$, such that y is in the set D." This statement probably assumes that G is a function with domain D. Also, the domain $[a, b] \times [c, d]$ is a rectangle, which is formally defined as the set of ordered pairs (x, y) in $\mathbf{R}^2$ such that $a \leq x \leq b$ and $c \leq y \leq d$.