Digest 6, Homework 6

(A compilation of emailed homework questions, answered around Wednesday, though changed to Sunday this week due to the quiz; I may or may not update this document on Monday.)

Question. (From a student:) When the question asks for us to determine if the following vector field is conservative or not, do we assume that the domain is simply connected and from there just use the cross partial test? I cannot determine if the domains are simply connected or not.

Answer. If the vector field F fails the cross partial test, then F is not conservative, and there is nothing you need to check about the domain. If the vector field F passes the cross partial test, and if F is defined on a simply connected domain D, then F is conservative on D.

A typical example of a vector field (e.g. Exercise 1) is a vector field F that is defined on all of $\mathbf{R}^3$. We also mentioned in class that $\mathbf{R}^3$ is simply connected. In summary, if F is defined on all of $\mathbf{R}^3$, then we have two options: (i) F passes the cross partial test and therefore F is conservative, or (ii) F fails the cross partial test and therefore F is not conservative.

The situation becomes more complicated when F is defined on other domains D. However, as discussed in class, as far as simple connectedness is concerned, I will only expect you to know the domains D that we listed in class.

Question. I am a little confused about exercise #5 in assignment 6. I know how to find the work done by F along the following path from $(1,0,0)$ to $(1,0,1)$, but the other bullet points are confusing me. What I have understand so far is that for the first bullet point you have to find the work done over the line segment that starts at $(1,0,0)$ and ends at $(1,0,1)$. But for the rest I have no idea how to approach them.

Answer. In the second bullet point, I specified a parametrization of a curve, so you want to find the work done along that specified curve. For the third bullet point, you need to find the work done over a slightly more complicated curve. If you want, you can find a parametrization of the curve; in particular, you would probably need to make two separate parameterizations, and then to add the contributions from both parameterizations.

Question. (From a student):

1. Geometrically, what is the difference between a line and flux integral? If a flux integral is the "amount of flow" through a parametrized curve, what is a line integral?

2. On a conservative closed loop, going counterclockwise, the normal for a flux integral going outwards is $(y', -x')$. When is the normal for a flux integral different from this vector? What is the normal for a parametrized curve that is not closed?

3. On a line integral, reversing the direction gives the negative of the original direction. Does this same rule apply to flux integrals?

Answer.

1. First of all, a line integral is just a generic term for the integral of any function on a line. I think you are asking: what is the difference between $\int_C F \cdot T ds$ (which is sometimes called the circulation integral) and $\int_C F \cdot e_n ds$ (which we refer to as the flux integral of a line in the plane). The first one can be interpreted as the amount of flow *along* a curve (i.e. parallel to the curve), and the second one can be interpreted, as you say, as the amount of flow *through* the curve (or perpendicular to it).

2. Once again, I think the terminology here is a bit mixed up. A vector field can be conservative (or not conservative). We never apply the term “conservative” to a closed loop. Anyway, for any given curve, there are two normal directions to choose from, and a normal vector can be specified for any curve. For example, if $s(t) = (x(t), y(t))$ is a curve, then it has tangent $s'(t) = (x'(t), y'(t))$, and there are two normal vectors that have the same length as $s'(t)$, namely $n(t) = (y'(t), -x'(t))$ and $-n(t)$. Note that $s'(t) \cdot n(t) = 0$ and $s'(t) \cdot (-n(t)) = 0$, so both $n(t)$ and $-n(t)$ are actually normal to $s'(t)$. In the case that s parameterizes a closed loop moving in the counterclockwise direction, then $n(t)$ as we have just defined it is the outward pointing normal vector, and we drew a picture about this in class.

3. As we discussed in class (and in Digest 5), reversing the direction of $\int_C F \cdot T ds$ multiplies the integral by -1, basically since the unit normal T is multiplied by -1. Also, as mentioned in class (and as you can readily verify in Exercise 10 on Homework 5), changing the direction of the parametrization does not change the value of $\int_C F \cdot e_n ds$, as long as the normal vector e_n points in the same direction, for the different parameterizations.