Digest 3, Homework 3

(A compilation of emailed homework questions, answered around Wednesday. Due to the day off this week, this digest will be sent around Monday, and then updated around Wednesday if more questions arrive.)

Question. [Exercise 6] Find the volume of the donut defined in spherical coordinates by $\rho \leq 2\sin\phi$.

(From a student:) I get to a step where i have to integrate $(\sin x)^4$. I was just wondering if I could skip this step by switching the order of integration. If this is not possible, how should I start to integrate it?

Answer. I would recommend using the double angle formulas to write $\sin^4(x)$ as a sum of other trigonometric functions.

Question. [Exercise 7] Suppose I want to design a structure of bounded height and with minimal moment of inertia. Specifically, suppose I have a region D in Euclidean space $\mathbf{R}^3$, and D lies between the planes $\{(x, y, z) \in \mathbf{R}^3 \colon z = 0\}$ and $\{(x, y, z) \in \mathbf{R}^3 \colon z = 1\}$. Suppose also that D has uniform density 1, and the mass of D is equal to 1. I then want to find the D with the smallest moment of inertia around the z axis. Which D should I use?

(From a student:) Wouldn't the region that has the least moment of inertia about the z axis simply be the point at the origin? If not, how can I set up this problem? The farthest I can get is $\iint (x^2 + y^2)dxdy))$ = moment of inertia. Then do I take the derivative of that to find a minimum?

Answer. This is kind of the right idea, but it is not entirely right. Note that the region D has uniform density 1, and its mass is equal to 1. So, if I take D to be a single point, then the mass of D is zero. So, this is not the correct region D. Do you see a way to enlarge the region D to get what you want? Lastly, I don't think taking a derivative is the right way to go here.

Question. [Exercise 10] ... Using integration in spherical coordinates, verify that $\iiint_{\mathbf{R}^3} p\, dV = 1$, so p actually represents a probability. Then, show that the probability of finding an electron at a distance greater than a from the origin is equal to $5/e^2 \approx .677$ ...

(From a student:) I'm not too sure how to how to tackle this problem, can you help set me in the right direction?

Answer. Well, the first thing I would do is to find the limits of integration with respect to spherical coordinates. How do we integrate over three-dimensional Euclidean space $\mathbf{R}^3$, using spherical coordinates?

Question. [Exercise 10] ... Using integration in spherical coordinates, verify that $\iiint_{\mathbf{R}^3} p\,dV = 1$, so p actually represents a probability. Then, show that the probability of finding an electron at a distance greater than a from the origin is equal to $5/e^2 \approx .677$ ...

(From a student:) I'm having trouble with Question 10. Could you point me in the right direction?

I'm integrating $e^{-2\rho/a}\rho^2 \sin\phi$. I first integrate with respect to ρ. In the end, I got the answer π. Could you give me a hint as to what part I'm doing wrong?

Answer. Most likely, you have integrated with respect to ρ incorrectly. For the inner integral, you need to integrate

$$\int_0^\infty e^{-2\rho/a}\rho^2\,d\rho = \lim_{L\to\infty}\int_0^L e^{-2\rho/a}\rho^2\,d\rho.$$

To integrate $\rho^2 e^{-2\rho/a}$ correctly on the interval $[0, L]$, you need to integrate by parts twice. Note that once you integrate with respect to ρ, the remaining parts of the integrand should just be a function of ϕ.

Question. [Example 3.32 from the notes, discussed in lecture] Let X and Y be random variables with joint density

$$p(x, y) = \begin{cases} \frac{1}{81}(2xy + 2x + y) & \text{, for } 0 \le x \le 3 \text{ and } 0 \le y \le 3 \\ 0 & \text{, otherwise} \end{cases}.$$

We calculate the probability that $X + Y < 3$. (Note that p satisfies $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p dA = 1$.) Let D be the set where $x + y < 3$, $x \ge 0$ and $y \ge 0$. Then $P(X + Y < 3) = \iint_D p dA$.

(From a student:) I am a bit confused about this example. How were you able to get $3 - x$ in the last probability example? In the beginning, it said $x+y < M$. How did the M become a 3?

Answer. Let D' be the set in the plane where $x + y < 3$. We define the probability $P(X + Y < 3)$ so that $P(X + Y < 3) = \iint_{D'} p dA$. Since p is nonzero only when $x \ge 0$ and $y \ge 0$, we have $\iint_{D'} p dA = \iint_D p dA$. Therefore, $P(X + Y < 3) = \iint_D p dA$. The limits of integration of D can then be described as the set of (x, y) where $0 \le x \le 3$ and $0 \le y \le 3-x$.

Question. (From a student:) How many homeworks are dropped in the computation of the course grade?

Answer. The lowest **two** homework grades are dropped. (Some versions of the syllabus had a typo on this point.)