Digest 9

(A compilation of emailed homework questions, answered around Wednesday.)

Question. (From a student): You assigned the div-curl = 0 identity for homework 9, and I'm not sure how I feel about it, as this identity is not as obvious to me as the curl/div-grad(f) models.

Is the correct idea to assume that it does not tell us anything? I know the math checks out (I actually have not went over it yet, but it should be as clean as the stokes'-divergence equality in R2) but I really don't know what to say in theory, so either a concise mathematical or theoretical explanation would be great!

Actually one more thing; is there anything particularly interesting about integrating curl(div(F)), since it seems to parallel the abstract motive of the div(curl(F))?

Also, for #9 for Assignment 9. When you say S(1) or S(z), are you saying map a sphere surface onto one variable?

Answer. One could try to explain $\mathrm{div}(\mathrm{curl}(F)) = 0$ for $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ as follows. The curl measures how much the vector field twists around. The divergence measures how much the vector field is spreading out. If you try to measure one after the other, you get zero. That is, if you are curling around, you also don't spread out. This is just an intuitive explanation, and it is not rigorous. A complete explanation of this identity is beyond the scope of this course. Suffice it to say that this identity is a special case of something much more general.

What is the meaning of $\mathrm{curl}(\mathrm{div}(F))$? If $F\colon \mathbf{R}^3 \to \mathbf{R}^3$, then this expression has no meaning, since the divergence is itself a function, but we can only take the curl of a vector field. If we instead consider $F\colon \mathbf{R}^2 \to \mathbf{R}^2$, then we still have a problem, since the divergence is a function, but we can only take the curl of vector fields.

Concerning the last question, I have adjusted the hint accordingly. When I say $S(1)$ I mean the sphere of radius 1 centered at the origin. Also, there was a typo in the statement of this exercise which has now been fixed.

Question. For the first problem, (part 1) is grad f a potential of a vector field? I am not sure I am understanding the question because can we only calculate the flux of a vector field but not of a function right?

Answer. $f\colon \mathbf{R}^2 \to \mathbf{R}$ is a function, so $\nabla f\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a vector field, so we can compute the flux of ∇f in the plane.

Question. Do we need to know the physics material in Chapter 18.3?

Answer. No. You need to know the Divergence Theorem, but you do not need to know the physics content of Chapter 18.3. More specifically, Maxwell's Laws will not be on the exam. The final covers everything from double integrals up through Green's, Stokes' and Divergence Theorems, but nothing more. Also, as mentioned previously, about two thirds of the final will be concerned with Green's, Stokes' and Divergence Theorems.

Question. (From a student): We know from Exercise 3 that, if $F = \mathrm{curl}(H)$ for a vector field H, then $\mathrm{div}(F) = \mathrm{div}(\mathrm{curl}(H)) = 0$. Does this mean that, if $\mathrm{div}(F) \neq 0$, then there does not exist a vector field H such that $F = \mathrm{curl}(H)$?

Answer. Yes. If $\mathrm{div}(F) \neq 0$ and if we have $F = \mathrm{curl}(H)$, then we would deduce from Exercise 3 that $\mathrm{div}(F) = \mathrm{div}(\mathrm{curl}(H)) = 0$. But this contradicts $\mathrm{div}(F) \neq 0$, so no such H exists.

Question. (From a student): We know from Exercise 3 that, if $F = \mathrm{curl}(H)$ for a vector field H, then $\mathrm{div}(F) = \mathrm{div}(\mathrm{curl}(H)) = 0$. What about the converse? That is, suppose $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a vector field such that $\mathrm{div}(F) = 0$. Is it true that there is another vector field $H\colon \mathbf{R}^3 \to \mathbf{R}^3$ such that $\mathrm{curl}(H) = F$?

Answer. (I should preface this question by saying that it deals with things outside the scope of the class, and of the final exam.)

Yes. This is related to the fact that, if $J\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a vector field with $\mathrm{curl}(J) = 0$, then there exists $f\colon \mathbf{R}^3 \to \mathbf{R}$ with $\nabla f = J$.

Write $H = (H_1, H_2, H_3)$, $F = (F_1, F_2, F_3)$. To see why our assertion is true, note that any function $f\colon \mathbf{R}^3 \to \mathbf{R}$ satisfies $\mathrm{curl}(\nabla f) = (0, 0, 0)$, so if such an H exists, then $\mathrm{curl}(H + \nabla f) = \mathrm{curl}(H) = F$. That is, if we choose f such that $\partial f/\partial z = -H_3$ (which we can by defining $f(x, y, z) = -\int_0^z H_3(x, y, z)dz$), then we cancel the third component of H. That is, we may assume at the outset that the third component of H is zero.

In summary, we have reduced to finding $H = (H_1, H_2, 0)$ such that $\mathrm{curl}(H) = F$. That is, we want to find H with $(-\partial H_2/\partial z, \partial H_1/\partial z, \partial H_2/\partial x - \partial H_1/\partial y) = (F_1, F_2, F_3)$. That is, we want to find H such that

$$\begin{cases} F_1 &= -\partial H_2/\partial z \\ F_2 &= \partial H_1/\partial z \\ F_3 &= \partial H_2/\partial x - \partial H_1/\partial y \end{cases}.$$

I claim that we can define H_1, H_2 by $H_1(x, y, z) = \int_0^z F_2(x, y, s)ds + h(x, y)$ and $H_2(x, y, z) = -\int_0^z F_1(x, y, s)ds$, for some function $h(x, y)$. Let's see why this definition works. First, note that $\partial H_1/\partial z = F_2(x, y, z)$, and $-\partial H_2/\partial z = F_1(x, y, z)$ by the Fundamental Theorem of

Calculus. So, it only remains to satisfy the third equality. Differentiating H_1, H_2, we get

$$\begin{aligned}
\partial H_2/\partial x - \partial H_1/\partial y &= -\int_0^z \frac{\partial}{\partial x} F_1(x,y,s)ds - \int_0^z \frac{\partial}{\partial y} F_2(x,y,s)ds - \frac{\partial}{\partial y} h(x,y) \\
&= \int_0^z \frac{\partial}{\partial z} F_3(x,y,s)ds - \frac{\partial}{\partial y} h(x,y) \qquad \text{, using } \mathrm{div}(F) = 0 \\
&= F_3(x,y,z) - F_3(x,y,0) - \frac{\partial}{\partial y} h(x,y) \qquad \text{, by the Fundamental Theorem of Calculus}
\end{aligned}$$

So, define $h(x,y) = -\int_0^y F_3(x,t,0)dt$. In summary, the following vector field $H\colon \mathbf{R}^3 \to \mathbf{R}^3$ satisfies $\mathrm{curl}(H)(x,y,z) = F(x,y,z)$.

$$H(x,y,z) = \left(\int_0^z F_2(x,y,s)ds - \int_0^y F_3(x,t,0)dt, \quad -\int_0^z F_1(x,y,s)ds, \quad 0 \right).$$

There is a caveat here. We have used that $F\colon \mathbf{R}^3 \to \mathbf{R}^3$ is defined on all of $\mathbf{R}^3$ and satisfies $\mathrm{div}(F) = 0$. If $F\colon D \to \mathbf{R}^3$ with $\mathrm{div}(F) = 0$ for certain domains D, then we could not in general find an $H\colon D \to \mathbf{R}^3$ with $\mathrm{curl}(H) = F$. For conservative vector fields, recall that if F is conservative on a simply connected domain, then $f = \nabla F$. In this case, we similarly need an assumption about the domain not having "holes." The statement would be something like: let S be any surface in D such that S has no boundary; if every such S can be shrunk down to a point in D, and if $F\colon D \to \mathbf{R}^3$ has $\mathrm{div}(F) = 0$, then there is an $H\colon D \to \mathbf{R}^3$ with $\mathrm{curl}(H) = F$.

To see why this assumption on the domain D is sometimes needed, consider the vector field we discussed in class, $F(x,y,z) = (x^2 + y^2 + z^3)^{-3/2}(x,y,z)$ which satisfies $\mathrm{div}(F) = 0$. Note that F is defined on the domain D which is $\mathbf{R}^3$ with the origin removed. Recall that D is simply connected! That is, any closed loop in D can be shrunk down to a point in D. However, the unit sphere in D cannot be shrunk down to a point in D. And there does not exist a vector field H with $\mathrm{curl}(H) = F$. To see this, suppose for the sake of contradiction that $F = \mathrm{curl}(H)$ for some vector field H. As we showed in class, if S is the sphere $x^2 + y^2 + z^2 = 1$, then $\iint_S F \cdot e_n dS = 4\pi$. However, if $F = \mathrm{curl}(H)$, then $\iint_S F \cdot e_n dS = \iint_S \mathrm{curl}(H) \cdot e_n dS = 0$ by Stokes' Theorem (since S has no boundary). Since we have achieved a contradiction, we conclude that F is not the curl of any vector field.

Good luck on the final!