Please provide complete and well-written solutions to the following exercises.

Assignment 7

Due November 13, at the beginning of class.

Exercise 1. Let $f(x, y, z) = x^2 + y^3 + z$. Compute the gradient $\nabla f(x, y, z)$. Find the linearization of f at the point $(a, b, c) = (1, 2, 3)$. Using this linearization and the approximation $f(x, y, z) \approx L(x, y, z)$, approximate the quantity $f(1.1, 1.9, 3.2)$.

Exercise 2. Let $f(x, y) = x^2y^3$. Compute the gradient $\nabla f(x, y)$. Then, find the tangent plane to the surface $z = f(x, y)$ at the point $(a, b) = (1, 2)$.

Exercise 3. Consider the unit sphere $x^2 + y^2 + z^2 = 1$. Let (a, b, c) be a point in the unit sphere. Find an equation for the tangent plane to the sphere at an arbitrary point (a, b, c).

Exercise 4. Let $f(x, y) = x^2+y^2$ and let $g(x, y) = x^2-y^2$. For any point (x, y) in the plane, we can plot the vector $\nabla f(x, y)$ in the plane, so that $\nabla f(x, y)$ has basepoint (x, y). Plotting the vector $\nabla f(x, y)$ in this way for many values of (x, y) allows us to visualize the gradient $\nabla f(x, y)$. For any x in the set $-2, -1, 0, 1, 2$, and for any y in the set $-2, -1, 0, 1, 2$, plot the gradient vector $\nabla f(x, y)$. Also, plot a few of the level curves of f. Then, in a separate drawing, plot the gradient vector $\nabla g(x, y)$, and also plot a few of the level curves of g.

Exercise 5. Let $f(x, y) = x^2 - y^2$. The Gaussian Curvature of the surface $z = f(x, y)$ at the point (a, b) is defined to be

$$\frac{f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2}{(1 + (f_x(a, b))^2 + (f_y(a, b))^2)^2}.$$

The Gaussian Curvature is supposed to measure how much the surface "curves around itself."

Compute the Gaussian Curvature of the saddle $f(x, y) = x^2 - y^2$ at $(a, b) = (0, 0)$. Then, compute the Gaussian curvature of the sphere of radius R: $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ for any point (a, b). Then, compute the Gaussian curvature of an arbitrary plane $f(x, y) = ax+by+c$, $a, b, c \in \mathbf{R}$. Finally, compute the Gaussian curvature of the cone $f(x, y) = \sqrt{x^2 + y^2}$. Which ones have positive, negative and zero curvature?

Exercise 6. The Griffith Observatory in Los Angeles has a large pendulum which is constantly swinging. The rotation of the earth about its axis causes the pendulum to change the apparent direction of motion of the pendulum.

If the pendulum were placed at the north pole, it would take 24 hours for the pendulum to return to its initial velocity as viewed from a person on the earth. This follows since, from the perspective of the sun, the pendulum always swings in essentially the same direction over a 24 hour period, whereas the earth has completed a full rotation about its axis over 24 hours. If the pendulum were placed at the equator, then from the perspective of the earth, the pendulum's apparent direction would never change.

Los Angeles has a latitude of about 34 degrees North. Compute the amount of time, in hours, that it takes for the Griffith Observatory pendulum to return to its initial velocity as viewd from a person in Los Angeles. (Hint: Let P be the plane which contains the equator of the earth. We can think of the velocity of the pendulum as a vector $v \in \mathbf{R}^3$, and we can then write $v = u + w$, where u is contained in P, and w is perpendicular to P. After 24 hours, we can think of the vector u as representing the velocity of a pendulum at the north pole, which has been rotated and then returned to its initial velocity. So, the number of rotations after 24 hours is $||u|| \,/\, ||v|| = \cos\theta$, where θ is the angle between your position on the earth and the north pole.)