Please provide complete and well-written solutions to the following exercises.

Assignment 6

No due date, but the quiz in Week 6 in the discussion section (on November 3rd or 5th) will be based upon this homework.

Exercise 1. Let $f(x,y) = x^2 + y^2$. Compute the partial derivatives: f_{xx}, f_{xy}, f_{yx}, f_{yy}.

Exercise 2. Let $f(u,v,w,x,y,z) = u^2/v + vxyz + \sin(xwv)$. Compute the partial derivatives: f_{uv}, f_{wz}, f_{xyz}.

Exercise 3. Consider the following function $f\colon \mathbf{R}^2 \to \mathbf{R}$.

$$f(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}, \qquad t > 0.$$

Show that f satisfies the **heat equation** (for one spatial dimension x):

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

The function f represents a single point of heat emanating through an infinite rod (the x-axis) as time passes (as t increases, $t \geq 0$). The heat equation roughly says that the rate of change of heat f at the point x and at time t is equal to the average difference between the current heat at x, and the neighbors of x. The quantity $\partial f/\partial t$ is the rate of change of heat, while the second derivative on the right is perhaps better understood using the second-difference quotient:

$$\partial^2 f/\partial x^2 = \lim_{h \to 0} \frac{f(x-h,t) - 2f(x,t) + f(x+h,t)}{h^2}.$$

Exercise 4. Consider a function $f(x,y,t)$ of three variables. The **heat equation** for two spatial dimensions x, y says

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

This equation can be interpreted in a similar way to the previous equation. As time goes to infinity, eventually the heat reaches an equilibrium, i.e. the heat does not change anymore, so $\partial f/\partial t$ goes to zero as $t \to \infty$. So, as $t \to \infty$, the heat equation will say that $f_{xx} + f_{yy} = 0$. We define $f_{xx} + f_{yy}$ to be the Laplacian Δ. That is, given a function $g(x,y)$ of two variables, define

$$\Delta g(x,y) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}.$$

A function g satisfying $\Delta g = 0$ is called **harmonic**. Understanding harmonic functions allows us to understand equilibrium configurations of heat. Show that the following functions are harmonic:

- $g(x,y) = x$.
- $g(x,y) = \tan^{-1}(y/x)$.
- $g(x,y) = \ln(x^2 + y^2)$.

Exercise 5. Consider the following function $f \colon \mathbf{R}^2 \to \mathbf{R}$.

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$$

Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$. (You will need to use the definition of the derivative itself, using limits.) So, it is not always true that $f_{xy} = f_{yx}$.