Digest 2

(A compilation of emailed homework questions, answered around Wednesday.)

Question. [Exercise 2] Let u, v be orthogonal vectors. Show that u, v satisfy the Pythagorean Theorem: $||u + v||^2 = ||u||^2 + ||v||^2$. Then, find vectors r, s such that $||r + s||^2 \neq ||r||^2 + ||s||^2$.

(From a student): Should we assume that the vectors u, v, r, and s all exist in $\mathbb{R}^2$ or $\mathbb{R}^3$? Or should we just work in $\mathbb{R}^n$ for any n?

Answer. This question is intentionally vague on this point. As you may find, it is possible to do first part of the problem without mentioning the dimension of the ambient space at all. For the second part, you are just asked to find vectors r, s, so you can choose them to be whatever you want (vectors in $\mathbf{R}^2$ might work, for example, or vectors in $\mathbf{R}^3$ might also work).

Question. [Exercise 6] Using the definition of the cross product, verify the following standard identities:

$$(1,0,0) \times (0,1,0) = (0,0,1), \quad (0,1,0) \times (0,0,1) = (1,0,0), \quad (0,0,1) \times (1,0,0) = (0,1,0).$$

$$(1,0,0) \times (1,0,0) = (0,1,0) \times (0,1,0) = (0,0,1) \times (0,0,1) = (0,0,0).$$

(From a student:) Does the phrase "the definition of the cross product" permit us to use the informal (determinant) definition, or should we stick strictly with the following?

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$$

Answer. Both formulas are the same, so using either formula will suffice. The determinant formula for the cross product may be a bit informal, but it agrees with the original definition.

Question. [Exercise 7] A tetrahedron has vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. This tetrahedron has four faces. Find the outward pointing unit normal vectors to each face of the tetrahedron.

(From a student:) How does one rigorously distinguish the interior and exterior of a closed surface in $\mathbb{R}^3$? Is there a symbolic way to distinguish an outward-pointing unit normal vector from an inward-pointing one, or will "drawing a diagram and looking at it" suffice for an explanation?

Answer. In general this can be a difficult question, so that in general there is no real "symbolic way" to find the interior/exterior (except maybe using the fact that the vectors $v, w, v \times w \in \mathbf{R}^3$ obey the right hand rule). So, yes, drawing a picture may just have to suffice sometimes. But for examples that we encounter, these tasks should never be impossible. Evidently this interior/exterior stuff is maybe the first hint that things can be quite different in multivariable calculus, compared to just single-variable calculus.

Informally, the interior region bounded by a closed surface are the points where you cannot "escape to infinity" without "crossing over" the surface. For example, consider the closed surface which is the unit sphere, $\{(x, y, z) \in \mathbf{R}^3 \colon x^2 + y^2 + z^2 = 1\}$. The interior region bounded by this surface could be defined as the closed unit ball, $\{(x, y, z) \in \mathbf{R}^3 \colon x^2+y^2+z^2 \leq 1\}$.

Similarly, all four faces of the tetrahedron are "enclosing" some points inside the tetrahedron. And we are asking for a vector which is normal to each face, where the vector is starting at the face, and pointing "inside" the tetrahedron. So in this particular, example, it should be clear what is meant by this interior region. For general examples, finding this interior region could become quite complicated, but I don't think I will ever give you an example where it is ridiculously complicated to find the interior region.

Finding the interior of a region can even be quite difficult in the plane $\mathbf{R}^2$. For example, if you have some closed curve in the plane, and it has many squiggles and such, can you really even "see" what the interior or exterior is? This can be difficult even to see visually. In fact, the mathematical version of "finding the interior of a curve" is a Theorem, known as the Jordan Curve Theorem. This theorem and it's generalization to higher dimensions are quite complicated to prove. However, we will probably not discuss these things very much in this class. (Interior/exterior will be important if you take 32B.)

Question. (From a student): I was wondering if you have a list of extra textbook questions we should do as practice?

Answer. I do not have such a list, but for extra practice, I would recommend doing a few of the easier ones, then focusing on the more challenging ones.

Question. (From a student): I was wondering if we could get the solutions to homework 2 before Tuesday because that's when our quiz is and that way we can rectify any mistake we're making before giving the quiz.

Answer. The homework solutions will be posted at the end of the week as usual. The quiz questions will be very similar to or identical to the homework questions. So, if I gave out the solutions before the quiz, that would be kind of like giving you the answers to a homework, and then having you turn in the homework. That is, it wouldn't really be testing anything meaningful.

However, as usual, you can ask specific questions via email or in office hours.