

32B: MULTIVARIABLE CALCULUS, SECOND QUARTER, WINTER 2015

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1. INTRODUCTION

As you have covered in your previous calculus classes, the subject of calculus has many applications. For example, calculus is very closely related to **probability**, which itself has applications to **statistics** and **algorithms**. For example, the first generation of Google's search technology came from ideas from probability theory. Within **physics**, differential equations often arise. In **economics**, optimization is often used to e.g. maximize profit margins. Also, the ideas of single variable calculus are developed and generalized within financial mathematics to e.g. **stochastic calculus**. Biology uses many ideas from calculus. Signal processing and **Fourier analysis** provide some nice applications within many areas of science. For example, our cell phones use Fourier analysis to compress voice signals.

Here are some applications that are specific to multivariable calculus. As we have seen in the single variable setting, **optimization** can be quite useful. Often we want to optimize some function of more than one variable, and doing so requires tools from multivariable calculus. Such optimization procedures appear in **economics** and **engineering**. The subject of **fluid mechanics** makes heavy use of multivariable calculus to describe the flow of fluids

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through different geometries and along different surfaces. Within physics, the theory of **general relativity** takes further many of the ideas that we will describe in this course, though this physical theory uses much more geometry than we will encounter. Within **medicine**, the mathematics of MRI and CT scans was discovered roughly forty years before these devices were invented. The relevant mathematics here uses both multivariable calculus and Fourier analysis.

2. DOUBLE INTEGRALS

Let $a < b$ be real numbers, and let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Recall that the integral $\int_a^b f(x)dx$ is approximated by **Riemann sums**. That is, if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

then $\int_a^b f(x)dx$, i.e. the area under the curve f , is approximated by the areas under boxes that approximate the function f . For example, if we evaluate f at the left endpoints of the boxes, we have

$$\int_a^b f(x)dx \approx \sum_{i=1}^n (x_i - x_{i-1})f(x_{i-1}).$$

The term on the right is the sum of areas of n boxes, where the i^{th} box has width $(x_i - x_{i-1})$ and height $f(x_{i-1})$. More specifically, we have the following limiting expression for the integral

$$\int_a^b f(x)dx = \lim_{(\max_{i=1}^n (x_i - x_{i-1})) \rightarrow 0} \sum_{i=1}^n (x_i - x_{i-1})f(x_{i-1}). \quad (*)$$

That is, no matter how we choose the points $a = x_0 < x_1 < \cdots < x_n = b$, as long as the quantity $(\max_{i=1}^n (x_i - x_{i-1}))$ goes to zero as $n \rightarrow \infty$, then the right side of $(*)$ approaches some real number. And that real number is called the **integral of f from a to b** , and it is denoted by the left side of $(*)$.

For a two-variable function, it turns out that we can adapt this procedure. Let $a < b$ and let $c < d$ be real numbers. Let $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a continuous function. We then split up the domain $[a, b] \times [c, d]$ into boxes. That is, let

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$c = y_0 < y_1 < \cdots < y_m = d.$$

We will then approximate the area under the function f by boxes. If $1 \leq i \leq n$ and if $1 \leq j \leq m$, then the box labelled (i, j) will have area $(x_i - x_{i-1})(y_j - y_{j-1})$. If we choose to evaluate f at the “lower-left” corner of each box, then this box will have height $f(x_{i-1}, y_{j-1})$. The volumes of these boxes will then form a **Riemann sum** as follows.

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y)dxdy \approx \sum_{j=1}^m \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})f(x_{i-1}, y_{j-1}).$$

More specifically, we have the following limiting expression for the integral

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y)dxdy = \lim_{\substack{(\max_{i=1}^n (x_i - x_{i-1})) \rightarrow 0 \\ (\max_{j=1}^m (y_j - y_{j-1})) \rightarrow 0}} \sum_{j=1}^m \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})f(x_{i-1}, y_{j-1}). \quad (*)$$

That is, no matter how we choose the points $a = x_0 < x_1 < \dots < x_n = b$ and $c = y_0 < y_1 < \dots < y_m = d$, as long as the quantities $(\max_{i=1}^n (x_i - x_{i-1}))$ and $(\max_{j=1}^m (y_j - y_{j-1}))$ go to zero as $n, m \rightarrow \infty$, then the right side of (*) approaches some real number. And that real number is called the **integral of f** on $[a, b] \times [c, d]$, and it is denoted by the left side of (*).

Remark 2.1. The same idea holds for an integral of $f(x, y)$ over a more general region D of the plane, but the Riemann sum is just more difficult to write. In the case that we integrate over a region D of the plane, we denote this integral by

$$\iint_D f dA.$$

Remark 2.2. In practice, computing a double integral is often done by computing two separate single-variable integrals.

Example 2.3. Let $a < b$ and let $c < d$ be real numbers. For any $a \leq x \leq b$ and $c \leq y \leq d$, let $f(x, y) = 1$. We evaluate the integral of f over $[a, b] \times [c, d]$ by first integrating in the x variable (while considering y fixed), and then by integrating in the y variable. That is,

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dx dy = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} dx \right) dy = \int_{y=c}^{y=d} (b - a) dy = (d - c)(b - a).$$

So, $\int_{y=c}^{y=d} \int_{x=a}^{x=b} dx dy$ gives the **area** of the rectangle $[a, b] \times [c, d]$. More generally, if D is a region in the plane, then the area of D is given by $\iint_D dA$.

Example 2.4. For any $0 \leq x \leq 1$ and $0 \leq y \leq 1$ let $f(x, y) = x + y$. We evaluate the integral of f by first integrating in the x variable (while considering y fixed), and then by integrating in the y variable. That is,

$$\begin{aligned} \int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x, y) dx dy &= \int_{y=0}^{y=1} \left(\int_{x=0}^{x=1} (x + y) dx \right) dy = \int_{y=0}^{y=1} [((1/2)x^2 + yx)|_{x=0}^{x=1}] dy \\ &= \int_{y=0}^{y=1} ((1/2) + y) dy = ((1/2)y + (1/2)y^2)|_{y=0}^{y=1} \\ &= (1/2) + (1/2) = 1. \end{aligned}$$

Remark 2.5. When we integrate on a rectangle $[a, b] \times [c, d]$, then we can change the order of integration. We can also change the order of integration when we integrate over general regions D , but we then need to be careful about describing the region D correctly.

Theorem 2.6 (Change of Order of Integration). *Let $a < b$ and let $c < d$ be real numbers. Let $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a continuous function. Let D denote the domain $D = [a, b] \times [c, d]$. Then*

$$\iint_D f dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dx dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) dy dx.$$

In order to change the order of integration of a more general region D , we need to make sure that the limits of the double integral define the region correctly.

Theorem 2.7 (Integration over General Regions D). *Let D be a region in the plane \mathbb{R}^2 . Let $a < b$ and let $c < d$ be real numbers.*

- Let $g_1: [a, b] \rightarrow \mathbb{R}$ and let $g_2: [a, b] \rightarrow \mathbb{R}$ be continuous functions. Suppose D is defined as the set of all (x, y) in the plane such that $a \leq x \leq b$ and such that $g_1(x) \leq y \leq g_2(x)$. Then we compute the integral $\iint_D f dA$ as follows.

$$\iint_D f dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx.$$

- Let $h_1: [c, d] \rightarrow \mathbb{R}$ and let $h_2: [c, d] \rightarrow \mathbb{R}$ be continuous functions. Suppose D is defined as the set of all (x, y) in the plane such that $c \leq y \leq d$ and such that $h_1(y) \leq x \leq h_2(y)$. Then we compute the integral $\iint_D f dA$ as follows.

$$\iint_D f dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x, y) dx dy.$$

Remark 2.8. Note that both items in this Theorem are the same, if we reverse the roles of x and y .

Example 2.9. Consider the region D defined as the set of all (x, y) in the plane such that $-1 \leq x \leq 1$ and such that $0 \leq y \leq \sqrt{1-x^2}$. Then D is a half of a disc. We will integrate D in two different ways, and we will get the same answer. Consider $f(x, y) = 2y$. Then

$$\begin{aligned} \iint_D f dA &= \int_{x=-1}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} 2y dy dx = \int_{x=-1}^{x=1} [(y^2)|_{y=0}^{y=\sqrt{1-x^2}}] dx \\ &= \int_{x=-1}^{x=1} (1-x^2) dx = [x - (1/3)x^3]_{x=-1}^{x=1} \\ &= 1 - (1/3) + 1 - (1/3) = 4/3. \end{aligned}$$

We now compute this integral by first integrating with respect to x , and then with respect to y . Note that D can be defined as the set of all (x, y) in the plane such that $0 \leq y \leq 1$ and such that $-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$. So,

$$\begin{aligned} \iint_D f dA &= \int_{y=0}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} 2y dx dy = \int_{y=0}^{y=1} (2y)(2\sqrt{1-y^2}) dy \\ &= \int_{u=1}^{u=0} (-2)\sqrt{u} du \quad , \text{ substituting } u = 1 - y^2 \\ &= \int_{u=0}^{u=1} 2\sqrt{u} du = (4/3)u^{3/2}|_{u=0}^{u=1} = 4/3. \end{aligned}$$

Remark 2.10. In the above example, note that since $f(x, y) \geq 0$ on D , $\iint_D f(x, y) dx dy$ gives the **volume** between the surfaces $z = f(x, y)$ and $z = 0$, over the region D .

Remark 2.11 (An Important Remark concerning the Square Root Function). In high school, you learned that the number 9 has two square roots 3 and -3 . When we write the symbol $\sqrt{9}$, we mean that the symbol $\sqrt{9}$ stands for the nonnegative number that, when squared, gives 9. So, $\sqrt{9} = 3$. So, do not confuse the **square root function** \sqrt{x} , which is a function from $[0, \infty)$ to $[0, \infty)$, with the **square roots of a number**. The square root function only has one value, and not two values.

Since the square root function applied to a positive number always gives a positive number, the following identity holds for any real number x

$$\sqrt{x^2} = |x|.$$

We used this identity in the above example. Note that, if x is negative, we have $\sqrt{x^2} \neq x$.

Example 2.12. Sometimes we are given a double integral, and we need to change the order of integration, since such a change may simplify the calculation of the integral. In such a case, drawing a picture of the region is helpful. For example, suppose we have the integral

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} f(x, y) dy dx.$$

By drawing a picture, we can see that the set of (x, y) in the plane with $0 \leq x \leq 1$ and with $x^2 \leq y \leq x$ is equal to the set of (x, y) in the plane with $0 \leq y \leq 1$ and with $x = y$ to $x = \sqrt{y}$. That is, we have

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} f(x, y) dy dx = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} f(x, y) dx dy.$$

The following properties are useful for computing various integrals. These properties should be familiar from the setting of single-variable integrals.

Proposition 2.13 (Properties of Integrals). *Let f and g be continuous functions in the plane, and let D be a region in the plane.*

- If c is a real number, then $\iint_D c f dA = c \iint_D f dA$.
- $\iint_D (f + g) dA = \iint_D f dA + \iint_D g dA$.
- $\iint_D (f - g) dA = \iint_D f dA - \iint_D g dA$.
- If $f(x, y) \geq 0$ for all (x, y) in D , then $\iint_D f dA \geq 0$.
- If $f(x, y) \geq g(x, y)$ for all (x, y) in D , then $\iint_D f dA \geq \iint_D g dA$.
- If D is the union of two nonoverlapping regions D_1 and D_2 , then

$$\iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA.$$

2.1. Polar Coordinates.

Definition 2.14. Let (x, y) be a point in the plane. That is, x and y are real numbers with $-\infty < x, y < \infty$. We define the polar coordinates (r, θ) of (x, y) with $0 \leq r < \infty$, $0 \leq \theta < 2\pi$, such that $r = \sqrt{x^2 + y^2}$, and such that θ satisfies $\theta \in [0, 2\pi)$ with

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Polar coordinates in the plane make certain expressions easier to understand. For example, the circle given by $x^2 + y^2 = 1$ can be equivalently written in polar coordinates as $r^2 = 1$, or just $r = 1$ since $r \geq 0$. Polar coordinates sometimes make integrals easier to compute as well, as we will see below. For now, let's get accustomed to polar coordinates.

Example 2.15. Plot the curve where $r = 1 - \cos \theta$.

We see that the following points (r, θ) are in the curve: $(0, 0)$, $(1, \pi/2)$, $(2, \pi)$, $(1, 3\pi/2)$. Also, from the increasing/decreasing and symmetry properties of $1 - \cos \theta$, we see that the curve should look like an apple, or cardioid. In this case, the Cartesian expression for the

curve is the rather complicated: $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$. To see this, note that $r^2 = r - r \cos \theta$, so $(r^2 + r \cos \theta)^2 = r^2$, so $(x^2 + y^2 + x)^2 = x^2 + y^2$, and so on.

Example 2.16. Plot the curve where $r = 4/(2 \cos \theta - \sin \theta)$.

Plotting this curve is a bit tricky to do directly, but it actually becomes simpler if we switch to Cartesian coordinates. We have $2r \cos \theta - r \sin \theta = 4$, so $2x - y = 4$, so we have the equation for a straight line, $y = 2x - 4$.

Exercise 2.17. Plot the function $r = \sqrt{\sin \theta}$ for $\theta \in (0, \pi)$. The end result should resemble a circle.

Exercise 2.18. Plot the function $r = 1 + \cos \theta$. The end result should resemble an apple, or a heart.

2.2. Integration in Polar Coordinates. If we are given a function $f(x, y)$ and the integral $\iint_D f(x, y) dx dy$ looks difficult to compute, sometimes it is easier to integrate the function $f(r \cos \theta, r \sin \theta)$. We can do this by the following theorem.

Theorem 2.19 (Changing from Cartesian to Polar Coordinates). *Let D be a region in the plane, and let $f(x, y)$ be a continuous function defined in the Cartesian coordinates (x, y) . Then*

$$\iint_D f(x, y) dx dy = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Remark 2.20. Note that there is an extra factor of r on the right side. We will justify this term later on in the course via the change of variables formula. For now, it suffices to note that the extra factor of r gives us the correct quantity πs^2 when we compute the area of a disc of radius s , as we show below.

Example 2.21. Let's compute a few integrals in polar coordinates. Consider the region D defined as the set of points (x, y) in the plane satisfying $x^2 + y^2 \leq 1$. Then the area of this disc of radius 1 is computed as follows.

$$\iint_D dx dy = \iint_D r dr d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r dr d\theta = \int_{\theta=0}^{\theta=2\pi} ((1/2)r^2)|_{r=0}^{r=1} d\theta = \int_{\theta=0}^{\theta=2\pi} (1/2) d\theta = \pi.$$

Similarly, let $s > 0$ and consider the region D defined as the set of points (x, y) in the plane satisfying $x^2 + y^2 \leq s^2$. Then the area of this disc of radius s is computed as follows.

$$\begin{aligned} \iint_D dx dy &= \iint_D r dr d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} r dr d\theta = \int_{\theta=0}^{\theta=2\pi} ((1/2)r^2)|_{r=0}^{r=s} d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} (1/2)s^2 d\theta = \pi s^2. \end{aligned}$$

Consider now the function $f(x, y) = x^2 + y^2$ integrated over D . We then have

$$\begin{aligned} \iint_D f(x, y) dx dy &= \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) r dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r^3 dr d\theta = \int_{\theta=0}^{\theta=2\pi} ((1/4)r^4)|_{r=0}^{r=1} d\theta = \int_{\theta=0}^{\theta=2\pi} (1/4) d\theta = \pi/2. \end{aligned}$$

Example 2.22. For the purposes of plotting functions, it is sometimes nice to consider negative values of r . We can make sense of negative values of r with the equalities $x = r \cos \theta$ and $y = r \sin \theta$. For example, consider the curve $r = \sin(2\theta)$, where $0 \leq \theta \leq 2\pi$. Then $r(\theta) > 0$ for $0 < \theta < \pi/2$, and $\pi < \theta < 3\pi/2$ and $r(\theta) < 0$ for $\pi/2 < \theta < \pi$ and $3\pi/2 < \theta < 2\pi$. Four (r, θ) points in the graph are then $(1, \pi/4)$, $(-1, 3\pi/4)$ (which in (x, y) coordinates is $(r \cos \theta, r \sin \theta) = (\sqrt{2}/2, -\sqrt{2}/2)$), $(1, 5\pi/4)$, and $(-1, 7\pi/4)$ (which in (x, y) coordinates is $(r \cos \theta, r \sin \theta) = (-\sqrt{2}/2, \sqrt{2}/2)$). The curve resembles a four-leaf flower.

The total area enclosed by the curve is four times the area of one of the “leaves” of the flower. That is, the total area enclosed by the curve is

$$\begin{aligned} 4 \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\sin(2\theta)} r dr d\theta &= 4 \int_{\theta=0}^{\theta=\pi/2} [(1/2)r^2]_{r=0}^{r=\sin(2\theta)} d\theta = 4 \int_{\theta=0}^{\theta=\pi/2} (1/2) \sin^2(2\theta) d\theta \\ &= \int_{\theta=0}^{\theta=\pi/2} (1 - \cos(4\theta)) d\theta = [\theta - (1/4) \sin(4\theta)]_{\theta=0}^{\theta=\pi/2} = \pi/2. \end{aligned}$$

It is possible to change the order of integration for this integral as well, though doing so is a bit complicated, since we have to be careful about the domain of the inverse sine function. For example, if $0 \leq \theta \leq \pi/4$, then $r = \sin(2\theta)$ can be written as $\theta = (1/2) \sin^{-1}(r)$. However, if $\pi/4 \leq \theta \leq \pi/2$, then $r = \sin(2\theta)$ means that $\theta = (\pi/2) - (1/2) \sin^{-1}(r)$. We therefore have

$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\sin(2\theta)} r dr d\theta = \int_{r=0}^{r=1} \int_{\theta=(1/2) \sin^{-1}(r)}^{\theta=(\pi/2)-(1/2) \sin^{-1}(r)} r d\theta dr.$$

3. TRIPLE INTEGRALS

Triple integrals are constructed in an analogous way to double integrals. We now sketch the idea.

Let $a < b$ and let $c < d$ and let $e < f$ be real numbers. Let $F: [a, b] \times [c, d] \times [e, f] \rightarrow \mathbb{R}$ be a continuous function. We then split up the domain $[a, b] \times [c, d] \times [e, f]$ into boxes. That is, let

$$\begin{aligned} a &= x_0 < x_1 < \cdots < x_n = b \\ c &= y_0 < y_1 < \cdots < y_m = d \\ e &= z_0 < z_1 < \cdots < z_\ell = f. \end{aligned}$$

We will then approximate the integral of f by boxes. If $1 \leq i \leq n$, if $1 \leq j \leq m$, and if $1 \leq k \leq \ell$, then the box labelled (i, j, k) will have volume $(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$. If we choose to evaluate f at the “lower-left” corner of each box, then this box will have height $f(x_{i-1}, y_{j-1}, z_{k-1})$. The volumes of these boxes multiplied by the value of f on the corner of the boxes will then form a **Riemann sum** as follows.

$$\begin{aligned} \int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} F(x, y, z) dx dy dz \\ \approx \sum_{k=1}^{\ell} \sum_{j=1}^m \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) F(x_{i-1}, y_{j-1}, z_{k-1}). \end{aligned}$$

More specifically, we have the following limiting expression for the integral

$$\begin{aligned} & \int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} F(x, y, z) dx dy dz \\ &= \lim_{\substack{(\max_{i=1}^n (x_i - x_{i-1})) \rightarrow 0 \\ (\max_{j=1}^m (y_j - y_{j-1})) \rightarrow 0 \\ (\max_{k=1}^\ell (z_k - z_{k-1})) \rightarrow 0}} \sum_{k=1}^{\ell} \sum_{j=1}^m \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) F(x_{i-1}, y_{j-1}, z_{k-1}). \quad (*) \end{aligned}$$

That is, no matter how we choose the points $a = x_0 < x_1 < \dots < x_n = b$, $c = y_0 < y_1 < \dots < y_m = d$, and $e = z_0 < z_1 < \dots < z_\ell = f$, as long as the quantities $(\max_{i=1}^n (x_i - x_{i-1}))$, $(\max_{j=1}^m (y_j - y_{j-1}))$ and $(\max_{k=1}^\ell (z_k - z_{k-1}))$ go to zero as $n, m, \ell \rightarrow \infty$, then the right side of (*) approaches some real number. And that real number is called the **integral of F** on $[a, b] \times [c, d] \times [e, f]$, and it is denoted by the left side of (*).

Remark 3.1. The same idea holds for an integral of $F(x, y, z)$ over a more general region D of Euclidean space, but the Riemann sum is just more difficult to write. In the case that we integrate over a region D of Euclidean space, we denote this integral by

$$\iiint_D F dV.$$

Remark 3.2. In practice, computing a triple integral is often done by computing three separate single-variable integrals.

Example 3.3. Let $a < b$ and let $c < d$ and let $e < f$ be real numbers. For any $a \leq x \leq b$, $c \leq y \leq d$, and $e \leq z \leq f$, let $F(x, y, z) = 1$. We evaluate the integral of F over $[a, b] \times [c, d] \times [e, f]$ by first integrating in the x variable (while considering y and z fixed), and then by integrating in the y variable (while considering z fixed), and then by integrating in the z variable. That is,

$$\begin{aligned} & \int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} F(x, y, z) dx dy dz \\ &= \int_{z=e}^{z=f} \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} dx \right) dy dz = \int_{z=e}^{z=f} \left(\int_{y=c}^{y=d} (b - a) dy \right) dz \\ &= \int_{z=e}^{z=f} (d - c)(b - a) dz = (f - e)(d - c)(b - a). \end{aligned}$$

So, $\int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} dx dy dz$ gives the **volume** of the box $[a, b] \times [c, d] \times [e, f]$. More generally, if D is a region in Euclidean space \mathbb{R}^3 , then the volume of D is given by $\iiint_D dV$.

Remark 3.4 (Visualizing Higher Dimensions). For a function $F(x, y)$ of two variables x, y , we often think of $F(x, y)$ as the height of the function F above the point (x, y) . Using this same visualization in three-dimensions doesn't work so well, since a function value $F(x, y, z)$ of three variables (x, y, z) would be interpreted as a fourth-dimension. Instead of thinking of $F(x, y, z)$ as a height, you can think of $F(x, y, z)$ being the color of a point (x, y, z) . For example, if $F(x, y, z)$ is large and positive, you can think of the point (x, y, z) as colored dark gray, and if $F(x, y, z)$ is very negative, you can think of the point (x, y, z) as colored light gray. So, a function on three-dimensional Euclidean space can be thought of as a grayscale coloring of Euclidean space.

Theorem 3.5 (Integration over General Regions D). Let D be a region in the Euclidean space \mathbb{R}^3 . Let $a < b$, $c < d$ and let $e < f$ be real numbers. Let F be a continuous function on \mathbb{R}^3 .

Let $g_1: [a, b] \rightarrow \mathbb{R}$ and let $g_2: [a, b] \rightarrow \mathbb{R}$ be continuous functions. Let $h_1: [a, b] \times [c, d] \rightarrow \mathbb{R}$ and let $h_2: [a, b] \times [c, d] \rightarrow \mathbb{R}$ be continuous functions. Suppose D is defined as the set of all (x, y, z) in \mathbb{R}^3 such that:

- $a \leq x \leq b$
- $g_1(x) \leq y \leq g_2(x)$
- $h_1(x, y) \leq z \leq h_2(x, y)$

Then we compute the integral $\iiint_D F dV$ as follows.

$$\iiint_D F dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} F(x, y, z) dz dy dx.$$

Remark 3.6. We can permute the roles of x, y, z in the above theorem as needed in applications.

Example 3.7. Find the volume of the cylinder C described as follows. The region C is the set of all (x, y, z) in Euclidean space such that $0 \leq z \leq 1$, and such that $x^2 + y^2 \leq 1$.

This cylinder has height 1 and radius 1, so we expect that C has volume π . Let's find the limits of integration and apply Theorem 3.5. C is defined as the set of points where $-1 \leq x \leq 1$, where $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$, and where $0 \leq z \leq 1$. So, the volume of C satisfies

$$\begin{aligned} \iiint_D dV &= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=0}^{z=1} dz dy dx \\ &= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dy dx = \int_{x=-1}^{x=1} 2\sqrt{1-x^2} dx \\ &= \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\sqrt{1-\sin^2 \theta} \cos \theta d\theta \quad , \text{ substituting } x = \sin \theta \\ &= \int_{\theta=-\pi/2}^{\theta=\pi/2} 2|\cos \theta| \cos \theta d\theta = \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\cos^2 \theta d\theta \quad , \text{ since } \cos \theta \geq 0 \text{ for } \theta \in [-\pi/2, \pi/2] \\ &= \int_{\theta=-\pi/2}^{\theta=\pi/2} (\cos(2\theta) + 1) d\theta = [(1/2)\sin(2\theta) + \theta]_{\theta=-\pi/2}^{\theta=\pi/2} = \pi. \end{aligned}$$

Example 3.8. Find the volume V of the region above the rectangle where $0 \leq x \leq 2$, $0 \leq y \leq 1$ and $z = 0$, bounded by the plane $y + z = 1$. Let's find the limits of integration and apply Theorem 3.5. C is defined as the set of points where $0 \leq x \leq 2$, where $0 \leq y \leq 1$, and where $0 \leq z \leq 1 - y$. So, the volume V of C satisfies

$$\begin{aligned} V &= \int_{x=0}^{x=2} \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} dz dy dx = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (1-y) dy dx \\ &= \int_{x=0}^{x=2} (y - (1/2)y^2)|_{y=0}^{y=1} dx = \int_{x=0}^{x=2} (1/2) dx = 1. \end{aligned}$$

Triple integrals satisfy the same properties as double integrals.

Proposition 3.9 (Properties of Integrals). Let f and g be continuous functions on \mathbb{R}^3 , and let D be a region in \mathbb{R}^3

- If c is a real number, then $\iiint_D cf dV = c \iiint_D f dV$.
- $\iiint_D (f + g) dV = \iiint_D f dV + \iiint_D g dV$.
- $\iiint_D (f - g) dV = \iiint_D f dV - \iiint_D g dV$.
- If $f(x, y, z) \geq 0$ for all (x, y, z) in D , then $\iiint_D f dV \geq 0$.
- If $f(x, y, z) \geq g(x, y, z)$ for all (x, y, z) in D , then

$$\iiint_D f dV \geq \iiint_D g dV.$$

- If D is the union of two nonoverlapping regions D_1 and D_2 , then

$$\iiint_D f dV = \iiint_{D_1} f dV + \iiint_{D_2} f dV.$$

3.1. Cylindrical and Spherical Coordinates.

3.1.1. Cylindrical Coordinates.

Definition 3.10. Let (x, y, z) be a point in the Euclidean space \mathbb{R}^3 . That is, x, y and z are real numbers with $-\infty < x, y, z < \infty$. We define the cylindrical coordinates (r, θ, z) of (x, y, z) with $0 \leq r < \infty$, $0 \leq \theta < 2\pi$, such that $r = \sqrt{x^2 + y^2}$, and such that θ satisfies $\theta \in [0, 2\pi)$ with

$$x = r \cos \theta, \quad y = r \sin \theta.$$

3.1.2. *Integration in Cylindrical Coordinates.* As we saw for polar coordinates, sometimes it is easier to integrate the a function by changing coordinates. We can do this by the following theorem.

Theorem 3.11 (Changing from Cartesian to Cylindrical Coordinates). Let D be a region in Euclidean space \mathbb{R}^3 , and let $f(x, y, z)$ be a continuous function defined in the Cartesian coordinates (x, y, z) . Then

$$\iiint_D f(x, y, z) dx dy dz = \iiint_D f(r \cos \theta, r \sin \theta, z) r dr d\theta dz.$$

Remark 3.12. Note that there is an extra factor of r on the right side. As in the case of polar coordinates, we will justify this term later on in the course via the change of variables formula. For now, note that it suffices to note that the extra factor of r gives us the correct quantity $\pi s^2 h$ when we compute the volume of a cylinder of radius s and height h , as we show below.

Example 3.13. Let's compute a few integrals in cylindrical coordinates. Let $s > 0$ and consider the region C defined as the set of points (x, y, z) in the plane satisfying $x^2 + y^2 \leq s^2$ and $0 \leq z \leq h$ with $h > 0$. We have computed the volume of C already using Cartesian coordinates, but let's see how much easier things are using cylindrical coordinates. The

volume of C is computed as follows

$$\begin{aligned} \iiint_C dx dy dz &= \iiint_C r dr d\theta dz = \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} r dr d\theta dz \\ &= \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} ((1/2)r^2)|_{r=0}^{r=s} d\theta dz = \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} (1/2)s^2 d\theta \\ &= \int_{z=0}^{z=h} \pi s^2 dz = \pi s^2 h. \end{aligned}$$

Consider now the circular cone D of radius $s > 0$ and height $h > 0$. That is, D is the set of points (x, y, z) in Euclidean space \mathbb{R}^3 such that $0 \leq z \leq h$ and such that $x^2 + y^2 \leq z^2 s^2 / h^2$. The volume of the circular cone D is given by

$$\begin{aligned} \iiint_D dx dy dz &= \iiint_D r dr d\theta dz = \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=zs/h} r dr d\theta dz \\ &= \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} [(1/2)r^2]_{r=0}^{r=zs/h} d\theta dz = \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} (1/2)z^2 s^2 / h^2 d\theta dz \\ &= \int_{z=0}^{z=h} \pi z^2 s^2 / h^2 dz = [(1/3)\pi z^3 s^2 / h^2]_{z=0}^{z=h} = (1/3)\pi h^3 s^2 / h^2 = (1/3)\pi s^2 h. \end{aligned}$$

3.1.3. Spherical Coordinates.

Definition 3.14. Let (x, y, z) be a point in the Euclidean space \mathbb{R}^3 . That is, x, y and z are real numbers with $-\infty < x, y, z < \infty$. We define the spherical coordinates (ρ, ϕ, θ) of (x, y, z) with $0 \leq \rho < \infty$, $0 \leq \phi \leq \pi$ and $0 \leq \theta < 2\pi$, such that $\rho = \sqrt{x^2 + y^2 + z^2}$, and such that θ satisfies $\theta \in [0, 2\pi)$ with

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}.$$

Also, ϕ is the angle that the vector (x, y, z) makes with the positive z axis. That is, $\phi \in [0, \pi]$ satisfies

$$\cos \phi = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \cdot (0, 0, 1) = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{\rho}.$$

To recover the Cartesian coordinates from the spherical coordinates, note that $\rho = \sqrt{r^2 + z^2}$, so $|r| = \sqrt{\rho^2 - z^2} = \sqrt{\rho^2 - \rho^2 \cos^2 \phi} = |\rho| |\sin \phi|$. Since $\rho \geq 0$ and $\sin \phi \geq 0$, we conclude that

$$r = \rho \sin \phi.$$

Therefore,

$$x = r \cos \theta = \rho \sin \phi \cos \theta, \quad y = r \sin \theta = \rho \sin \phi \sin \theta.$$

And, as we already mentioned, $z = \rho \cos \phi$.

3.1.4. Integration in Spherical Coordinates.

Theorem 3.15 (Changing from Cartesian to Spherical Coordinates). *Let D be a region in Euclidean space \mathbb{R}^3 , and let $f(x, y, z)$ be a continuous function defined in the Cartesian coordinates (x, y, z) . Then*

$$\iiint_D f(x, y, z) dx dy dz = \iiint_D f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta.$$

Remark 3.16. Note that there is an extra factor of $\rho^2 \sin \phi$ on the right side. As in the case of cylindrical coordinates, we will justify this term later on in the course via the change of variables formula. For now, it suffices to note that the extra factor on $\rho^2 \sin \phi$ gives us the correct quantity $(4/3)\pi s^3$ when we compute the volume of a ball of radius s .

Example 3.17. Let's compute the volume of a ball in Euclidean space of radius $s > 0$. Let B denote the region of all (x, y, z) in Euclidean space \mathbb{R}^3 such that $x^2 + y^2 + z^2 \leq s^2$. Then B is equivalently described as the set of (ρ, ϕ, θ) such that $0 \leq \rho \leq s$, $0 \leq \phi \leq \pi$ and such that $0 \leq \theta \leq 2\pi$. Then the volume of B is given by

$$\begin{aligned} \iiint_B dV &= \iiint_B \rho^2 \sin \phi d\rho d\phi d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=s} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} [(1/3)\rho^3 \sin \phi]_{\rho=0}^{\rho=s} d\phi d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} (1/3)s^3 \sin \phi d\phi d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} [(1/3)s^3(-\cos \phi)]_{\phi=0}^{\phi=\pi} d\theta = \int_{\theta=0}^{\theta=2\pi} (2/3)s^3 d\theta = (4/3)\pi s^3. \end{aligned}$$

3.2. Applications of Triple Integrals.

Definition 3.18 (Average Value). Let D be a region in Euclidean space \mathbb{R}^3 . Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function. Then the **average value** of f on D is defined as

$$\frac{\iiint_D f dV}{\iiint_D dV}.$$

Definition 3.19 (Mass). Let D be a region in Euclidean space \mathbb{R}^3 . For each point (x, y, z) in D , let $f(x, y, z)$ denote the density of the object D . Then the **mass** M of D is defined as

$$M = \iiint_D f(x, y, z) dx dy dz.$$

Definition 3.20 (Center of Mass). Let D be a region in Euclidean space \mathbb{R}^3 . For each point (x, y, z) in D , let $f(x, y, z)$ denote the density of the object D . Then the **center of mass** of D is defined as the following point in Euclidean space \mathbb{R}^3 .

$$\left(\frac{1}{M} \iiint_D x f(x, y, z) dx dy dz, \frac{1}{M} \iiint_D y f(x, y, z) dx dy dz, \frac{1}{M} \iiint_D z f(x, y, z) dx dy dz \right).$$

Definition 3.21 (Moment of inertia). Let D be a region in Euclidean space \mathbb{R}^3 . For each point (x, y, z) in D , let $f(x, y, z)$ denote the density of the object D . Then the **moment of inertia** of D about the z axis is defined as

$$\iiint_D (x^2 + y^2) f(x, y, z) dx dy dz.$$

The **moment of inertia** of D about the y axis is defined as

$$\iiint_D (x^2 + z^2) f(x, y, z) dx dy dz.$$

The **moment of inertia** of D about the x axis is defined as

$$\iiint_D (y^2 + z^2) f(x, y, z) dx dy dz.$$

The moment of inertia measures how difficult it is to rotate an object about an axis.

Example 3.22. Let D denote cylinder of radius $s > 0$ and height 1 centered at the origin, of uniform density 1. That is, D is the set of all points (x, y, z) with $x^2 + y^2 \leq s^2$ and with $-1/2 \leq z \leq 1/2$. Also, $f(x, y, z) = 1$ for all (x, y, z) in D . Then the mass M of D can be computed using cylindrical coordinates as follows

$$\begin{aligned} M &= \iiint_D f(x, y, z) dx dy dz = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} r dz dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} r dr d\theta = \int_{\theta=0}^{\theta=2\pi} (1/2)s^2 d\theta = \pi s^2. \end{aligned}$$

We now calculate the center of mass of the cylinder, again using cylindrical coordinates

$$\begin{aligned} \iiint_D x f(x, y, z) dx dy dz &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} r^2 \cos \theta dz dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} r^2 \cos \theta dr d\theta = \int_{\theta=0}^{\theta=2\pi} (1/3)s^3 \cos \theta d\theta = 0. \end{aligned}$$

$$\begin{aligned} \iiint_D y f(x, y, z) dx dy dz &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} r^2 \sin \theta dz dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} r^2 \sin \theta dr d\theta = \int_{\theta=0}^{\theta=2\pi} (1/3)s^3 \sin \theta d\theta = 0. \end{aligned}$$

$$\begin{aligned} \iiint_D z f(x, y, z) dx dy dz &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} z r dz dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} (0) r dr d\theta = 0. \end{aligned}$$

In conclusion, the center of mass of D is $(0, 0, 0)$. We now compute the moments of inertia of D about the coordinate axes. The moment of inertia of D about the z axis is given by

$$\begin{aligned} \iiint_D (x^2 + y^2) f(x, y, z) dx dy dz &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} r^3 dz dr d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} r^3 dr d\theta = \int_{\theta=0}^{\theta=2\pi} (1/4)s^4 d\theta = \pi s^4 / 2. \end{aligned}$$

The moment of inertia of D about the x axis is given by

$$\begin{aligned}
\iiint_D (y^2 + z^2)f(x, y, z)dxdydz &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} (r^3 \sin^2 \theta + rz^2)dzdrd\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} [r^3 z \sin^2 \theta + (1/3)rz^3]_{z=-1/2}^{z=1/2} dzdrd\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} (r^3 \sin^2 \theta + (1/12)r)drd\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} [(1/4)r^4 \sin^2 \theta + (1/24)r^2]_{r=0}^{r=s} d\theta = \int_{\theta=0}^{\theta=2\pi} [(1/4)s^4 \sin^2 \theta + (1/24)s^2]d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} [(1/8)s^4(1 - \cos(2\theta)) + (1/24)s^2]d\theta = [(1/8)s^4(\theta - (1/2)\sin(2\theta)) + (1/24)s^2\theta]_{\theta=0}^{\theta=2\pi} \\
&= (1/4)\pi s^2(s^2 + 1/3).
\end{aligned}$$

Similarly, the moment of inertia of D about the y axis is given by

$$\begin{aligned}
\iiint_D (x^2 + z^2)f(x, y, z)dxdydz &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} \int_{z=-1/2}^{z=1/2} (r^3 \cos^2 \theta + rz^2)dzdrd\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} [r^3 z \cos^2 \theta + (1/3)rz^3]_{z=-1/2}^{z=1/2} dzdrd\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} (r^3 \cos^2 \theta + (1/12)r)drd\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=s} [(1/4)r^4 \cos^2 \theta + (1/24)r^2]_{r=0}^{r=s} d\theta = \int_{\theta=0}^{\theta=2\pi} [(1/4)s^4 \cos^2 \theta + (1/24)s^2]d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} [(1/8)s^4(1 + \cos(2\theta)) + (1/24)s^2]d\theta \\
&= [(1/8)s^4(\theta + (1/2)\sin(2\theta)) + (1/24)s^2\theta]_{\theta=0}^{\theta=2\pi} = (1/4)\pi s^2(s^2 + 1/3).
\end{aligned}$$

Example 3.23. Let's calculate the moment of inertia about the z axis of the region D where $x^2 + y^2 + z^2 \leq 1$, $x \geq 0$, $y \geq 0$, and $z \geq 0$. Using spherical coordinates, we compute

$$\begin{aligned}
&\iiint_D (x^2 + y^2)dxdydz \\
&= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/2} (\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta)\rho^2 \sin \phi d\phi d\theta d\rho \\
&= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/2} \rho^4 \sin^3 \phi d\phi d\theta d\rho \\
&= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\pi/2} [\rho^4(-\cos \phi + (1/3)\cos^3 \phi)]_{\phi=0}^{\phi=\pi/2} d\theta d\rho \\
&= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\pi/2} \rho^4(2/3)d\theta d\rho = (\pi/3) \int_{\rho=0}^{\rho=1} \rho^4 d\rho = \pi/15.
\end{aligned}$$

Exercise 3.24. Suppose I want to design a structure of bounded height and with minimal moment of inertia. Specifically, suppose I have a region D in Euclidean space \mathbb{R}^3 , and D lies between the planes $\{(x, y, z) \in \mathbb{R}^3: z = 0\}$ and $\{(x, y, z) \in \mathbb{R}^3: z = 1\}$. Suppose also that D has uniform density, and the mass of D is equal to 1. I then want to find the D with the smallest moment of inertia around the z axis. Which D should I use?

3.2.1. Probability Theory.

Definition 3.25. A **random variable** X is a function $X: [0, 1] \rightarrow \mathbb{R}$. We interpret the value of X as the outcome of some experiment or measurement where the value of X is not known in advance. A random variable **has density** $p(x)$ if there exists a function $p: \mathbb{R} \rightarrow [0, \infty)$ with $\int_{-\infty}^{\infty} p(x)dx = 1$ such that, for any real numbers $a \leq b$, we have

$$P(a \leq X \leq b) = \int_a^b p(x)dx.$$

Here we read $P(a \leq X \leq b)$ as the probability that X lies between a and b .

Definition 3.26. Let X be a random variable with density p . The **expected value** (or average value) of X is defined as

$$\int_{-\infty}^{\infty} xp(x)dx.$$

Example 3.27. A random variable X is called a **standard Gaussian** if X has density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. The function $xe^{-x^2/2}$ is odd, so the expected value of X is $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2}dx = 0$.

Definition 3.28. Let X and Y be random variables. We say that X and Y **have joint density** $p(x, y)$ if there exists a function $p: \mathbb{R}^2 \rightarrow [0, \infty)$ with $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)dxdy = 1$ such that, and any real numbers $a \leq b$ and $c \leq d$, we have

$$P(a \leq X \leq b; c \leq Y \leq d) = \int_{x=a}^{x=b} \int_{y=c}^{y=d} p(x, y)dxdy.$$

Here we read $P(a \leq X \leq b; c \leq Y \leq d)$ as the joint probability that X and Y satisfy $a \leq X \leq b$ and that $c \leq Y \leq d$.

Definition 3.29. Let X and Y be random variables with joint density p . Let M be a real number. Let D be the set $\{(x, y): x + y < M\}$. Then

$$P(X + Y < M) = \iint_D pdA.$$

More generally, if $g(x, y)$ is a function, if M is a real number, and if D is the set where $g(x, y) \leq M$, then

$$P(g(X, Y) \leq M) = \iint_D pdA.$$

Example 3.30. Suppose X and Y are random variables with joint density $p(x, y)$. Let D be the unit disc, where $x^2 + y^2 \leq 1$. Then

$$P(X^2 + Y^2 \leq 1) = \iint_D pdA.$$

Definition 3.31. If X, Y have joint density $p(x, y)$, then we define the **expected value** of $X + Y$ to be

$$\int_{\mathbb{R}^2} (x + y)p(x, y)dxdy.$$

Example 3.32. Let X and Y be random variables with joint density

$$p(x, y) = \begin{cases} \frac{1}{81}(2xy + 2x + y) & , \text{ for } 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 3 \\ 0 & , \text{ otherwise} \end{cases}.$$

We calculate the probability that $X + Y < 3$. (Note that p satisfies $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p dA = 1$.) Let D be the set where $x + y < 3$, $x \geq 0$ and $y \geq 0$. Then

$$\begin{aligned} P(X + Y < 3) &= \iint_D p dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} p(x, y) dy dx = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} \frac{1}{81}(2xy + 2x + y) dy dx \\ &= \int_{x=0}^{x=3} \frac{1}{81}(xy^2 + 2xy + (1/2)y^2)|_{y=0}^{y=3-x} dx \\ &= \int_{x=0}^{x=3} \frac{1}{81}(x(3-x)^2 + 2x(3-x) + (1/2)(3-x)^2) dx \\ &= \int_{x=0}^{x=3} \frac{1}{81}(x^3 - (15/2)x^2 + 12x + 9/2) dx = \frac{1}{81}((1/4)x^4 - (5/2)x^3 + 6x^2 + (9/2)x)|_{x=0}^{x=3} \\ &= (1/81)((81/4) - (5/2)3^3 + (2)3^3 + (1/2)3^3) = (1/4) - (5/6) + (2/3) + (1/6) = 1/4. \end{aligned}$$

3.2.2. *The Heat Equation.* We think of introducing a heated object D into a (two-dimensional) room, and allowing the heat to dissipate by diffusion

Suppose we have a function $u(x, y, t)$ where (x, y) are points in the plane, and $t \geq 0$ represents time. Suppose at time $t = 0$, we have $u(x, y, 0) = 1$ if (x, y) lies in D , and $u(x, y, 0) = 0$ otherwise. Suppose u satisfies the following partial differential equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad (*)$$

So, we have some initial distribution of heat in the region D , and we want to see how the heat dissipates as time goes on. The equation (*) says that the change in the value of $u(x, y, t)$ is equal to the average difference of u with its neighbors. Specifically, the right hand side can be expressed as the symmetric difference quotient.

$$\lim_{h \rightarrow 0} \frac{u(x+h, y, t) + u(x-h, y, t) + u(x, y+h, t) + u(x, y-h, t) - 4u(x, y, t)}{h^2}.$$

We then claim that we can solve for u with the following formula.

$$u(x, y, t) := \frac{1}{4\pi t} \iint_D e^{-\frac{(v-x)^2 + (w-y)^2}{4t}} dv dw.$$

We can verify that u solves equation (*) by differentiating under the integral sign as follows.

$$\frac{\partial}{\partial t} u = \frac{1}{16\pi t} \iint_D t^{-2}((v-x)^2 + (w-y)^2) e^{-\frac{(v-x)^2 + (w-y)^2}{4t}} dv dw - \frac{1}{4\pi t^2} \iint_D e^{-\frac{(v-x)^2 + (w-y)^2}{4t}} dv dw$$

$$\frac{\partial}{\partial x} u = \frac{1}{4\pi t} \iint_D \frac{(v-x)}{2t} e^{-\frac{(v-x)^2 + (w-y)^2}{4t}} dv dw.$$

$$\frac{\partial^2}{\partial x^2} u = \frac{1}{4\pi t} \iint_D \left(\frac{(v-x)^2}{4t^2} - \frac{1}{2t} \right) e^{-\frac{(v-x)^2+(w-y)^2}{4t}} dv dw.$$

$$\frac{\partial^2}{\partial y^2} u = \frac{1}{4\pi t} \iint_D \left(\frac{(w-y)^2}{4t^2} - \frac{1}{2t} \right) e^{-\frac{(v-x)^2+(w-y)^2}{4t}} dv dw.$$

We can also use a probabilistic interpretation of this formula: $u(x, y, t)$ is the probability that a particle, with starting position (x, y) , undergoing Brownian motion from time zero to time t will be present in the set D at time t . So, the heat equation can be interpreted as diffusion of a particle. This is one of many manifestations of equivalent concepts in calculus and probability.

Remark 3.33. Similar formulas work for one or three space dimensions.

Remark 3.34. The Black-Scholes equation from finance theory is a variant of the one-dimensional heat equation, and its solution can be written in a form similar to the one written above.

4. CHANGE OF VARIABLES

We can finally examine a formula that justifies the way we changed our integrals between Cartesian, Cylindrical, and Spherical coordinates. This formula is known as the change of variables formula. Before examining this formula, let's recall how we change variables for one-dimensional integrals.

Theorem 4.1 (Change of Variables/ Substitution Rule, Dimension 1). *Let $a < b$ and let $c < d$ be real numbers. Let $g: [a, b] \rightarrow [c, d]$ be a differentiable function such that the derivative g' is continuous. Let $f: [c, d] \rightarrow \mathbb{R}$ be a continuous function. Then*

$$\int_{g(a)}^{g(b)} f(x) dx = \int_a^b f(g(y)) g'(y) dy.$$

We want to look at an analogue of this for higher dimensions. Within this analogous result, we will multiply on the right side by some derivatives, but we will also need to use an absolute value sign. Before stating the change of variables for two dimensions, we need two definitions.

Definition 4.2 (Jacobian determinant). Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a function. That is, for each (x, y) in the plane \mathbb{R}^2 , we write $G(x, y) = (a(x, y), b(x, y))$, so that $a(x, y)$ and $b(x, y)$ are real numbers. For any (x, y) in \mathbb{R}^2 , we define the **Jacobian determinant** $\text{Jac}(G)(x, y)$ as the following real number:

$$\text{Jac}(G)(x, y) = \det \begin{pmatrix} \frac{\partial}{\partial x} a(x, y) & \frac{\partial}{\partial y} a(x, y) \\ \frac{\partial}{\partial x} b(x, y) & \frac{\partial}{\partial y} b(x, y) \end{pmatrix} = \frac{\partial}{\partial x} a(x, y) \cdot \frac{\partial}{\partial y} b(x, y) - \frac{\partial}{\partial y} a(x, y) \cdot \frac{\partial}{\partial x} b(x, y).$$

We need to make a few technical assumptions about the functions that will change variables. In this course, we will not need to pay much attention to these assumptions, but they are good to know. One such assumption is the following.

Definition 4.3. Let D and S be regions in the plane \mathbb{R}^2 . Let $G: D \rightarrow S$ be a function. We say that G is a **one-to-one correspondence** if, for every (a, b) in S , there exists exactly one (x, y) in D such that $G(x, y) = (a, b)$.

Theorem 4.4 (Change of Variables, Dimension 2). Let D and S be regions in the plane \mathbb{R}^2 . Let $G: D \rightarrow S$ be a map that is also a one-to-one correspondence. Assume that G is differentiable and all of its partial derivatives are continuous. Let $f: S \rightarrow \mathbb{R}$ be a continuous function. For any (x, y) in D , assume that the Jacobian determinant $\text{Jac}(G)(x, y)$ is not zero. Then we have the following equality

$$\iint_S f(a, b) da db = \iint_D f(G(x, y)) |\text{Jac}(G)(x, y)| dx dy.$$

If we write $S = G(D)$, then we can equivalently write

$$\iint_{G(D)} f(a, b) da db = \iint_D f(G(x, y)) |\text{Jac}(G)(x, y)| dx dy.$$

Remark 4.5. Note that we take the absolute value of the Jacobian on the right side, unlike the case of one-variable.

Example 4.6. Let's see what the change of variables formula says about areas. That is, let S be some region in the plane \mathbb{R}^2 , let w be a vector in the plane \mathbb{R}^2 , and let D be the region in the plane \mathbb{R}^2 defined by $D = \{y + w: y \in S\}$. Then D is equal to the region S , translated by the vector $w = (w_1, w_2)$. Define $G: D \rightarrow S$ by $G(x, y) = (x - w_1, y - w_2) = (x, y) - w$. Then G translates D in the direction $-w$. We say that G is a **translation**. We write the components of G as (a, b) so that $a(x, y) = x - w_1$ and $b(x, y) = y - w_2$. We then compute

$$\text{Jac}(G)(x, y) = \frac{\partial}{\partial x} a(x, y) \cdot \frac{\partial}{\partial y} b(x, y) - \frac{\partial}{\partial y} a(x, y) \cdot \frac{\partial}{\partial x} b(x, y) = 1 - 0 = 1.$$

We therefore have

$$\iint_S da db = \iint_D dx dy.$$

That is, the area of S is invariant under any translation of S .

Example 4.7. We now consider a **rotation** G . Let $\theta \in \mathbb{R}$ and define

$$G(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then G rotates (x, y) counterclockwise around the origin by the angle θ . We compute

$$\begin{aligned} \text{Jac}(G)(x, y) &= \frac{\partial}{\partial x} (x \cos \theta - y \sin \theta) \cdot \frac{\partial}{\partial y} (x \sin \theta + y \cos \theta) - \frac{\partial}{\partial y} (x \cos \theta - y \sin \theta) \cdot \frac{\partial}{\partial x} (x \sin \theta + y \cos \theta) \\ &= \cos^2 \theta + \sin^2 \theta = 1. \end{aligned}$$

So, if we define $S = \{G(y): y \in D\}$, we get

$$\iint_S da db = \iint_D dx dy.$$

That is, the area of S is invariant under any rotation of S .

Example 4.8. Let's now investigate a **dilation**. Let $\lambda > 0$. Let D be a region in the plane \mathbb{R}^2 . Let S be the region defined by $S = \{(\lambda x, \lambda y): (x, y) \in D\}$. Then S is equal to the region D , dilated by λ . Define $G: D \rightarrow S$ by $G(x, y) = (\lambda x, \lambda y)$. Then G dilates D by λ .

We say that G is a **dilation**. We write the components of G as (a, b) so that $a(x, y) = \lambda x$ and $b(x, y) = \lambda y$. We then compute

$$\text{Jac}(G)(x, y) = \frac{\partial}{\partial x}a(x, y) \cdot \frac{\partial}{\partial y}b(x, y) - \frac{\partial}{\partial y}a(x, y) \cdot \frac{\partial}{\partial x}b(x, y) = \lambda^2.$$

We therefore have

$$\iint_S dadb = \lambda^2 \iint_D dx dy.$$

That is, when we take a two-dimensional region D and dilate it by λ to get S , then the area of S is λ^2 times the original area of D . This observation agrees with our formula for the area s^2 of a square of side length s . It also agrees with our formula for the area πs^2 of a disc of radius s .

Example 4.9. Let's use this Theorem to justify our change of variables from Cartesian to polar coordinates, Theorem 2.19. For any (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$, define

$$G(r, \theta) = (r \cos \theta, r \sin \theta).$$

We then compute

$$\begin{aligned} \text{Jac}(G)(r, \theta) &= \frac{\partial}{\partial r}(r \cos \theta) \cdot \frac{\partial}{\partial \theta}(r \sin \theta) - \frac{\partial}{\partial \theta}(r \cos \theta) \cdot \frac{\partial}{\partial r}(r \sin \theta) \\ &= (\cos \theta)(r \sin \theta) - (-r \sin \theta)(\sin \theta) = r \cos^2 \theta + r \sin^2 \theta = r. \end{aligned}$$

Since $r \geq 0$, we have $|\text{Jac}(G)(r, \theta)| = |r| = r$ for all polar coordinates (r, θ) . We therefore get

$$\iint_{G(D)} f(x, y) dx dy = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Example 4.10. Let's compute the following integral using the change of variables formula.

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \sqrt{x+y} (y-2x)^2 dy dx.$$

We use the transformation $u(x, y) = x + y$ and $v(x, y) = y - 2x$. That is, we define $G(x, y) = (u(x, y), v(x, y)) = (x + y, y - 2x)$. Let D denote the region of integration, where $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$. Then D is a right triangle, where $x \geq 0$, $y \geq 0$, and $y \leq 1 - x$. In order to determine the limits of integration for $G(D)$, we solve for x and y in terms of u and v . That is, we have $u - v = 3x$, and $2u + v = 3y$, so $x = (1/3)(u - v)$ and $y = (1/3)(2u + v)$. So, $x \geq 0$, means $v \leq u$; $y \geq 0$ means $v \geq -2u$; and $y \leq 1 - x$ means $2u + v \leq 3 - u + v$, so $u \leq 1$. The lines $v = u$ and $v = -2u$ intersect at the point $(u, v) = (0, 0)$. So, the new limits of integration can be described as $0 \leq u \leq 1$, and $-2u \leq v \leq u$.

Let's compute the Jacobian determinant of G .

$$\text{Jac}(G)(x, y) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} = 1 \cdot 1 - (-2) \cdot (1) = 1 + 2 = 3.$$

Finally, using the change of variables formula, we have

$$\begin{aligned}
\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \sqrt{x+y} (y-2x)^2 dy dx &= \frac{1}{3} \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \sqrt{x+y} (y-2x)^2 3 dy dx \\
&= \frac{1}{3} \iint_D \sqrt{x+y} (y-2x)^2 |\text{Jac}(G)(x,y)| dy dx = \frac{1}{3} \iint_{G(D)} u^{1/2} v^2 dv du \\
&= \frac{1}{3} \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2 dv du = \frac{1}{3} \int_{u=0}^{u=1} [v^3/3]_{v=-2u}^{v=u} u^{1/2} du = \int_{u=0}^{u=1} u^3 u^{1/2} du = (2/9).
\end{aligned}$$

There is also a change of variables theorem for dimension 3, though the formula for the Jacobian has more terms.

Definition 4.11 (Jacobian determinant). Let $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a function. That is, for each (x, y, z) in the plane \mathbb{R}^3 , we write $G(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z))$, so that $a(x, y, z)$, $b(x, y, z)$ and $c(x, y, z)$ are real numbers. For any (x, y, z) in \mathbb{R}^3 , we define the **Jacobian determinant** $\text{Jac}(G)(x, y, z)$ as the following real number:

$$\begin{aligned}
\text{Jac}(G)(x, y, z) &= \det \begin{pmatrix} \frac{\partial}{\partial x} a(x, y, z) & \frac{\partial}{\partial y} a(x, y, z) & \frac{\partial}{\partial z} a(x, y, z) \\ \frac{\partial}{\partial x} b(x, y, z) & \frac{\partial}{\partial y} b(x, y, z) & \frac{\partial}{\partial z} b(x, y, z) \\ \frac{\partial}{\partial x} c(x, y, z) & \frac{\partial}{\partial y} c(x, y, z) & \frac{\partial}{\partial z} c(x, y, z) \end{pmatrix} \\
&= \frac{\partial}{\partial x} a(x, y, z) \cdot \frac{\partial}{\partial y} b(x, y, z) \frac{\partial}{\partial z} c(x, y, z) - \frac{\partial}{\partial x} a(x, y, z) \cdot \frac{\partial}{\partial z} b(x, y, z) \frac{\partial}{\partial y} c(x, y, z) \\
&\quad - \frac{\partial}{\partial y} a(x, y, z) \cdot \frac{\partial}{\partial x} b(x, y, z) \frac{\partial}{\partial z} c(x, y, z) + \frac{\partial}{\partial y} a(x, y, z) \cdot \frac{\partial}{\partial z} b(x, y, z) \frac{\partial}{\partial x} c(x, y, z) \\
&\quad + \frac{\partial}{\partial z} a(x, y, z) \cdot \frac{\partial}{\partial x} b(x, y, z) \frac{\partial}{\partial y} c(x, y, z) - \frac{\partial}{\partial z} a(x, y, z) \cdot \frac{\partial}{\partial y} b(x, y, z) \frac{\partial}{\partial x} c(x, y, z).
\end{aligned}$$

Definition 4.12. Let D and S be regions in Euclidean space \mathbb{R}^3 . Let $G: D \rightarrow S$ be a function. We say that G is a **one-to-one correspondence** if, for every (a, b, c) in S , there exists exactly one (x, y, z) in D such that $G(x, y, z) = (a, b, c)$.

Theorem 4.13 (Change of Variables, Dimension 3). Let D and S be regions in Euclidean space \mathbb{R}^3 . Let $G: D \rightarrow S$ be a map that is also a one-to-one correspondence. Assume that G is differentiable and all of its partial derivatives are continuous. Let $f: S \rightarrow \mathbb{R}$ be a continuous function. For any (x, y, z) in D , assume that the Jacobian determinant $\text{Jac}(G)(x, y, z)$ is not zero. Then we have the following equality

$$\iiint_S f(a, b, c) da db dc = \iiint_D f(G(x, y, z)) |\text{Jac}(G)(x, y, z)| dx dy dz.$$

If we write $S = G(D)$, then we can equivalently write

$$\iiint_{G(D)} f(a, b, c) da db dc = \iiint_D f(G(x, y, z)) |\text{Jac}(G)(x, y, z)| dx dy dz.$$

Remark 4.14. Note that we take the absolute value of the Jacobian on the right side, unlike the case of one-variable.

Example 4.15. Let's see what the change of variables formula says about volumes. Let S be some region in Euclidean space \mathbb{R}^3 , let w be a vector in \mathbb{R}^3 , and let D be the region

in Euclidean space \mathbb{R}^3 defined by $D = \{y + w : y \in S\}$. Then D is equal to the region S , translated by the vector $w = (w_1, w_2, w_3)$. Define $G: D \rightarrow S$ by $G(x, y, z) = (x - w_1, y - w_2, z - w_3) = (x, y, z) - w$. Then G translates D in the direction $-w$. We say that G is a **translation**. We write the components of G as (a, b, c) so that $a(x, y, z) = x - w_1$, $b(x, y, z) = y - w_2$ and $c(x, y, z) = z - w_3$. We then compute

$$\begin{aligned} \text{Jac}(G)(x, y, z) &= \frac{\partial}{\partial x}a(x, y, z) \cdot \frac{\partial}{\partial y}b(x, y, z) \frac{\partial}{\partial z}c(x, y, z) - \frac{\partial}{\partial x}a(x, y, z) \cdot \frac{\partial}{\partial z}b(x, y, z) \frac{\partial}{\partial y}c(x, y, z) \\ &\quad - \frac{\partial}{\partial y}a(x, y, z) \cdot \frac{\partial}{\partial x}b(x, y, z) \frac{\partial}{\partial z}c(x, y, z) + \frac{\partial}{\partial y}a(x, y, z) \cdot \frac{\partial}{\partial z}b(x, y, z) \frac{\partial}{\partial x}c(x, y, z) \\ &\quad + \frac{\partial}{\partial z}a(x, y, z) \cdot \frac{\partial}{\partial x}b(x, y, z) \frac{\partial}{\partial y}c(x, y, z) - \frac{\partial}{\partial z}a(x, y, z) \cdot \frac{\partial}{\partial y}b(x, y, z) \frac{\partial}{\partial x}c(x, y, z) \\ &= 1 - 0 - 0 + 0 + 0 - 0 = 1. \end{aligned}$$

We therefore have

$$\iiint_S dadbdc = \iiint_D dx dy dz.$$

That is, the volume of S is invariant under any translation of S .

Let's now investigate a **dilation**. Let $\lambda > 0$. Let D be a region in Euclidean space \mathbb{R}^3 . Let S be the region defined by $S = \{\lambda y : y \in R\}$. Then S is equal to the region D , dilated by λ . Define $G: D \rightarrow S$ by $G(x, y, z) = (\lambda x, \lambda y, \lambda z)$. Then G dilates D by λ . We say that G is a **dilation**. We write the components of G as (a, b, c) so that $a(x, y, z) = \lambda x$, $b(x, y, z) = \lambda y$ and $c(x, y, z) = \lambda z$. We then compute

$$\begin{aligned} \text{Jac}(G)(x, y, z) &= \frac{\partial}{\partial x}a(x, y, z) \cdot \frac{\partial}{\partial y}b(x, y, z) \frac{\partial}{\partial z}c(x, y, z) - \frac{\partial}{\partial x}a(x, y, z) \cdot \frac{\partial}{\partial z}b(x, y, z) \frac{\partial}{\partial y}c(x, y, z) \\ &\quad - \frac{\partial}{\partial y}a(x, y, z) \cdot \frac{\partial}{\partial x}b(x, y, z) \frac{\partial}{\partial z}c(x, y, z) + \frac{\partial}{\partial y}a(x, y, z) \cdot \frac{\partial}{\partial z}b(x, y, z) \frac{\partial}{\partial x}c(x, y, z) \\ &\quad + \frac{\partial}{\partial z}a(x, y, z) \cdot \frac{\partial}{\partial x}b(x, y, z) \frac{\partial}{\partial y}c(x, y, z) - \frac{\partial}{\partial z}a(x, y, z) \cdot \frac{\partial}{\partial y}b(x, y, z) \frac{\partial}{\partial x}c(x, y, z) \\ &= \lambda^3 - 0 - 0 + 0 + 0 - 0 = \lambda^3. \end{aligned}$$

We therefore have

$$\iiint_S dadbdc = \lambda^3 \iiint_D dx dy dz.$$

That is, when we take a three-dimensional region D and dilate it by λ to get S , then the volume of S is λ^3 times the original volume of D . This observation agrees with our formula for the volume s^3 of a cube of side length s . It also agrees with our formula for the volume $(4/3)\pi s^3$ of a ball of radius s .

Example 4.16. Let's use this Theorem to justify our change of variables from Cartesian to spherical coordinates, Theorem 3.15. For any (ρ, ϕ, θ) with $\rho \geq 0$, $0 \leq \phi \leq \pi$ and $0 \leq \theta < 2\pi$ define

$$G(\rho, \phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi).$$

We then compute

$$\begin{aligned} \text{Jac}(G)(\rho, \phi, \theta) &= \frac{\partial}{\partial \rho}(\rho \sin \phi \cos \theta) \cdot \frac{\partial}{\partial \phi}(\rho \sin \phi \sin \theta) \frac{\partial}{\partial \theta}(\rho \cos \phi) - \frac{\partial}{\partial \rho}(\rho \sin \phi \cos \theta) \cdot \frac{\partial}{\partial \theta}(\rho \sin \phi \sin \theta) \frac{\partial}{\partial \phi}(\rho \cos \phi) \\ &\quad - \frac{\partial}{\partial \phi}(\rho \sin \phi \cos \theta) \cdot \frac{\partial}{\partial \rho}(\rho \sin \phi \sin \theta) \frac{\partial}{\partial \theta}(\rho \cos \phi) + \frac{\partial}{\partial \phi}(\rho \sin \phi \cos \theta) \cdot \frac{\partial}{\partial \theta}(\rho \sin \phi \sin \theta) \frac{\partial}{\partial \rho}(\rho \cos \phi) \\ &\quad + \frac{\partial}{\partial \theta}(\rho \sin \phi \cos \theta) \cdot \frac{\partial}{\partial \rho}(\rho \sin \phi \sin \theta) \frac{\partial}{\partial \phi}(\rho \cos \phi) - \frac{\partial}{\partial \theta}(\rho \sin \phi \cos \theta) \cdot \frac{\partial}{\partial \phi}(\rho \sin \phi \sin \theta) \frac{\partial}{\partial \rho}(\rho \cos \phi) \end{aligned}$$

That is,

$$\begin{aligned} \text{Jac}(G)(\rho, \phi, \theta) &= (\sin \phi \cos \theta)(\rho \cos \phi \sin \theta)(0) - (\sin \phi \cos \theta)(\rho \sin \phi \cos \theta)(-\rho \sin \phi) \\ &\quad - (\rho \cos \phi \cos \theta)(\sin \phi \sin \theta)(0) + (\rho \cos \phi \cos \theta)(\rho \sin \phi \cos \theta)(\cos \phi) \\ &\quad + (-\rho \sin \phi \sin \theta)(\sin \phi \sin \theta)(-\rho \sin \phi) - (-\rho \sin \phi \sin \theta)(\rho \cos \phi \sin \theta)(\cos \phi). \\ &= \rho^2 \sin^3 \phi \cos^2 \theta + \rho^2 \cos^2 \phi \sin \phi \cos^2 \theta + \rho^2 \sin^3 \phi \sin^2 \theta + \rho^2 \cos^2 \phi \sin \phi \sin^2 \theta \\ &= \rho^2 \sin \phi \cos^2 \theta + \rho^2 \sin \phi \sin^2 \theta = \rho^2 \sin \phi. \end{aligned}$$

Since $\rho \geq 0$ and $0 \leq \phi \leq \pi$, we have $\sin \phi \geq 0$, so that $|\text{Jac}(G)(\rho, \phi, \theta)| = |\rho^2 \sin \phi| = \rho^2 \sin \phi$ for all spherical coordinates (ρ, ϕ, θ) . We therefore get

$$\iiint_{G(D)} f(x, y, z) dx dy dz = \iiint_D f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta.$$

Exercise 4.17. Let $d, e, f > 0$ be constants. Consider the ellipsoid, defined as the set of (a, b, c) in Euclidean space \mathbb{R}^3 such that $a^2/d^2 + b^2/e^2 + c^2/f^2 \leq 1$. Compute the volume of this ellipsoid in terms of d, e, f . (Hint: let D denote the ball $x^2 + y^2 + z^2 \leq 1$, then define $G(x, y, z) = (dx, ey, fz) = (a(x, y, z), b(x, y, z), c(x, y, z))$.)

5. VECTOR FIELDS

There are many ways to think of a vector field; one is the velocity flow of a fluid; or the strength and direction of some electromagnetic field. Generally, we can think of a vector field as just an assignment of a vector to each point in Euclidean space.

In the previous course, we considered parameterized curves in the plane and in three-dimensional space. A parameterized curve assigns for each real t a point $s(t)$ in either the plane or in three-dimensional space. Specifically, a parameterized curved in the plane is a function $s: \mathbb{R} \rightarrow \mathbb{R}^2$. We then write $s(t) = (x(t), y(t))$, so that $x(t)$ and $y(t)$ are real numbers for each real number t . If we interpret $s(t)$ as the position of a moving object at time t , then the derivative $(d/dt)s(t) = ((d/dt)x(t), (d/dt)y(t))$ is the velocity of s at time t . We can then interpret $(d/dt)s(t)$ as a vector, compute its length, and so on. Specifically, $(d/dt)s(t)$ is the velocity of the object at the time t and position $s(t)$.

A vector field is a kind of generalization of the velocity of a parameterized curve. The velocity of the parameterized curve assigns a vector to each point along the curve. A vector field is a function that assigns a vector to every point. For example, if we want to look at the flow of air or some other fluid, we can assign to every point the velocity vector of the air or fluid. For another example, to every point in space we can assign the direction and strength of a gravitational or electromagnetic field. (The vector points in the direction of the flow of the field, and the length of the vector gives the strength of the field.) For points

near the surface of the earth, the gravitational field is a vector that is pointing towards the center of the earth with length around 9.8, using the units of meters per second squared.

Definition 5.1 (Vector, Dimension 2). A **vector** in the plane is a point (x, y) in the plane \mathbb{R}^2 . We draw the vector (x, y) as an arrow with basepoint the origin, and endpoint (x, y) in the plane \mathbb{R}^2 . The **dot product** of two vectors (x, y) and (a, b) is the quantity

$$(a, b) \cdot (x, y) = ax + by.$$

The **length** $\|(x, y)\|$ of the vector (x, y) is the quantity

$$\|(x, y)\| = \sqrt{x^2 + y^2} = \sqrt{(x, y) \cdot (x, y)}.$$

Definition 5.2 (Vector Field, Dimension 2). A **vector field** in the plane is a function from the plane \mathbb{R}^2 to the plane \mathbb{R}^2 . Specifically, a vector field is a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. For each point (x, y) in the plane \mathbb{R}^2 , we think of $f(x, y)$ as a vector in \mathbb{R}^2 with basepoint at (x, y) .

Example 5.3. Let (x, y) be a variable point in the plane \mathbb{R}^2 . Define $g(x, y) = (x, y)$. Then g is a vector field. If we draw the vector $g(x, y)$ so that its basepoint is (x, y) , then $g(x, y)$ is always pointing away from the origin.

Example 5.4. Let (x, y) be a variable point in the plane \mathbb{R}^2 . Define $g(x, y) = (-y, x)$. Then g is a vector field. If we draw the vector $g(x, y)$ so that its basepoint is (x, y) , then $g(x, y)$ is always pointing in a direction perpendicular to the origin. If we look at many such vectors $g(x, y)$, then g looks like a whirlpool.

Definition 5.5 (Vector, Dimension 3). A **vector** in the Euclidean space is a point (x, y, z) in Euclidean space \mathbb{R}^3 . We draw the vector (x, y, z) as an arrow with basepoint the origin, and endpoint (x, y, z) in Euclidean space \mathbb{R}^3 . The **dot product** of two vectors (x, y, z) and (a, b, c) is the quantity

$$(a, b, c) \cdot (x, y, z) = ax + by + cz.$$

The **length** $\|(x, y, z)\|$ of the vector (x, y, z) is the quantity

$$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(x, y, z) \cdot (x, y, z)}.$$

Definition 5.6 (Vector Field, Dimension 3). A **vector field** in Euclidean space is a function from Euclidean space \mathbb{R}^3 to Euclidean space \mathbb{R}^3 . Specifically, a vector field is a function $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. For each point (x, y, z) in Euclidean space \mathbb{R}^3 , we think of $f(x, y, z)$ as a vector in \mathbb{R}^3 with basepoint (x, y, z) .

Example 5.7. Let (x, y, z) be a variable point in Euclidean space \mathbb{R}^3 . Define $g(x, y, z) = (x, y, z)$. Then g is a vector field. If we draw the vector $g(x, y, z)$ so that its basepoint is (x, y, z) , then $g(x, y, z)$ is always pointing away from the origin.

Example 5.8. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real-valued function. So, $f(x, y, z)$ is a real number for each (x, y, z) in Euclidean space \mathbb{R}^3 . Recall that the gradient of f is defined as the following vector in Euclidean space \mathbb{R}^3

$$\nabla f(x, y, z) = \left(\frac{\partial}{\partial x} f(x, y, z), \frac{\partial}{\partial y} f(x, y, z), \frac{\partial}{\partial z} f(x, y, z) \right).$$

Then $\nabla f(x, y, z)$ is a vector field.

5.1. Line Integrals. There are several different ways to integrate vector fields. Let's begin with one way which resembles things we have done before. Suppose C is a path in Euclidean space \mathbb{R}^3 . That is, C is a continuously differentiable curve, and we can find some parametrization $s: [0, 1] \rightarrow \mathbb{R}^3$ that is continuously differentiable and which does not overlap onto itself. (For each $t_1, t_2 \in [0, 1]$ with $t_1 \neq t_2$, we have $s(t_1) \neq s(t_2)$.) We write

$$s(t) = (x(t), y(t), z(t)).$$

Here t is a real number, and for each real number t , we have $x(t), y(t)$ and $z(t)$ which are all real numbers. In the previous class, we defined the arc length C , or the arc length of s from 0 to 1 by

$$\int_C ds = \int_{t=0}^{t=1} \|(d/dt)s(t)\| dt = \int_{t=0}^{t=1} \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2 + \left(\frac{d}{dt}z(t)\right)^2}.$$

We need to assume that s does not overlap onto itself so that we do not double-count points in C .

For example, the circle in the plane is the set of (x, y) in the plane with $x^2 + y^2 = 1$, and we parameterize this circle by $s(t) = (\cos(t), \sin(t))$, where $t \in [0, 2\pi)$. We then know that $s'(t) = (-\sin(t), \cos(t))$, so $\|s'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$, so

$$\int_C ds = \int_{t=0}^{t=2\pi} dt = 2\pi.$$

However, if we parameterize the circle by $s(t) = (\cos(t), \sin(t))$ with $t \in [0, 4\pi]$, then s will travel twice along the circle, so the arc length of s will not be equal to the arc length of C . The point is, we need to be careful about parametrizing C .

We can similarly integrate functions along paths, if we just multiply the value of some function within the integrand as follows.

Definition 5.9 (Line Integrals of Functions). Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real-valued function on Euclidean space \mathbb{R}^3 . Let C be a continuously differentiable curve in Euclidean space \mathbb{R}^3 . Let $a < b$ be real numbers. Suppose $s: [a, b] \rightarrow \mathbb{R}^3$ is a parametrization of the curve C . That is, C is equal to the set $\{s(t): t \in [a, b]\}$, and s is a continuously differentiable function that does not overlap onto itself. (For all $t_1, t_2 \in [a, b]$ with $t_1 \neq t_2$, we have $s(t_1) \neq s(t_2)$.) We then define the **line integral** of f on C as the following one-variable integral

$$\int_C f ds = \int_{t=a}^{t=b} f(s(t)) \|(d/dt)s(t)\| dt.$$

Remark 5.10. The integral $\int_C f ds$ does not depend on the parametrization of the curve C in the following sense. Suppose $s: [a, b] \rightarrow \mathbb{R}^3$ and $\tilde{s}: [c, d] \rightarrow \mathbb{R}^3$ are two equivalent parameterizations such that $t: [c, d] \rightarrow [a, b]$ satisfies $s(t(r)) = \tilde{s}(r)$. (That is, we can reparameterize s to get \tilde{s} .) Then the chain rule says that $s'(t(r))t'(r) = \tilde{s}'(r)$. So, changing variables, we have

$$\int_{t=a}^{t=b} f(s(t)) \|s'(t)\| dt = \int_{r=c}^{r=d} f(s(t(r))) \|s'(t(r))t'(r)\| dr = \int_{r=c}^{r=d} f(\tilde{s}(r)) \|\tilde{s}'(r)\| dr.$$

Remark 5.11. Note that if we reverse the direction of the parametrization, then the integral does not change sign. For example, if $f(x, y, z) = 1$ for all points (x, y, z) , and if $s(t) = (t, 0, 0)$ parametrizes the straight line C where $0 \leq t \leq 1$, then

$$\int_C f ds = \int_{t=0}^{t=1} \|(1, 0, 0)\| dt = \int_{t=0}^{t=1} dt = 1.$$

And if $\tilde{s}(t) = (1 - t, 0, 0)$ denotes the parametrization of C in the reverse direction, where $0 \leq t \leq 1$, then

$$\int_C f d\tilde{s} = \int_{t=0}^{t=1} \|(-1, 0, 0)\| dt = \int_{t=0}^{t=1} dt = 1.$$

So, $\int_C f ds = \int_C f d\tilde{s}$.

Example 5.12. We integrate the function $f(x, y) = x^2$ on the circle C where $x^2 + y^2 = 1$. As before, we parameterize the circle by $s(t) = (\cos(t), \sin(t))$, where $t \in [0, 2\pi)$. We then know that $s'(t) = (-\sin(t), \cos(t))$, so $\|s'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$, so

$$\int_C f ds = \int_{t=0}^{t=2\pi} f(s(t)) \|s'(t)\| dt = \int_{t=0}^{t=2\pi} \cos^2(t) dt = \int_{t=0}^{t=2\pi} (1/2)(1 + \cos(2t)) dt = \pi.$$

We will integrate vector fields along paths, and even along surfaces. These integrals have physical meaning in applications, which we will try to describe along the way.

Definition 5.13 (Line Integral of a Vector Field along a Curve). Let $a < b$ be real numbers. Let $s: [a, b] \rightarrow \mathbb{R}^3$ be a parametrization of the curve C . Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. For each $c \in C$, let $T(c)$ denote the unit tangent vector to C at c . We define the **line integral** of F over s by

$$\int_C F \cdot T ds = \int_{t=a}^{t=b} (F(s(t))) \cdot \frac{(d/dt)s(t)}{\|(d/dt)s(t)\|} \|(d/dt)s(t)\| dt = \int_{t=a}^{t=b} (F(s(t))) \cdot ((d/dt)s(t)) dt.$$

Alternatively, we can write the expression for $\int_C F \cdot T ds$ in coordinates. Write

$$F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)), \quad s(t) = (x(t), y(t), z(t)).$$

We then have

$$\int_C F \cdot T ds = \int_{t=a}^{t=b} \left(F_1(x(t), y(t), z(t)) \frac{d}{dt} x(t) + F_2(x(t), y(t), z(t)) \frac{d}{dt} y(t) + F_3(x(t), y(t), z(t)) \frac{d}{dt} z(t) \right) dt.$$

If F is a vector field representing a force field, then $\int F \cdot T ds$ is called the **work** of F along s . If F is a vector field representing the velocity vectors of a fluid flow, then $\int F \cdot T ds$ is called the **flow integral** of F along s .

Example 5.14. Let $F(x, y, z) = (0, 0, z)$ be a vector field, and consider the helix C which is parameterized by $s(t) = (\cos t, \sin t, t)$ where $t \in [0, 2\pi]$. Then

$$\begin{aligned} \int_C F \cdot T ds &= \int_{t=0}^{t=2\pi} F(s(t)) \cdot s'(t) dt = \int_{t=0}^{t=2\pi} (0, 0, t) \cdot (-\sin t, \cos t, 1) dt \\ &= \int_{t=0}^{t=2\pi} t dt = (1/2)(2\pi)^2 = 2\pi^2. \end{aligned}$$

Remark 5.15. As in Remark 5.10, re-parameterizing the curve s does not change the value of $\int_C F \cdot T ds$, if we maintain the orientation of the parametrization. That is, if we reverse the direction of the parametrization, then the line integral of a vector field *does* change sign. Continuing the previous example, if $\tilde{s}(t) = (\cos(2\pi - t), \sin(2\pi - t), 2\pi - t)$ where $t \in [0, 2\pi]$, then

$$\begin{aligned} \int_C F \cdot T d\tilde{s} &= \int_{t=0}^{t=2\pi} F(\tilde{s}(t)) \cdot \tilde{s}'(t) dt = \int_{t=0}^{t=2\pi} (0, 0, 2\pi - t) \cdot (\sin(2\pi - t), -\cos(2\pi - t), -1) dt \\ &= \int_{t=0}^{t=2\pi} (t - 2\pi) dt = - \int_{\tilde{t}=0}^{\tilde{t}=2\pi} \tilde{t} d\tilde{t} = -2\pi^2. \end{aligned}$$

Here we changed variables, setting $\tilde{t} = 2\pi - t$. So, $\int_C F \cdot T ds = - \int_C F \cdot T d\tilde{s}$.

Definition 5.16 (Flux Integral in the Plane). Let C be a closed curve in the plane \mathbb{R}^2 . Let $a < b$ be real numbers. Let $s: [a, b] \rightarrow \mathbb{R}^2$ be a parametrization of the curve C . Since C is closed, we have $s(a) = s(b)$. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field. For each $c \in C$, let $e_n(c)$ denote the outward-pointing unit normal vector to C . We define the **flux** of F across C by

$$\oint_C F \cdot e_n ds.$$

Let us find a coordinate expression for the flux. For each (x, y) in the plane \mathbb{R}^2 , write $F(x, y) = (F_1(x, y), F_2(x, y))$. Also write $s(t) = (x(t), y(t))$. Recall that $s'(t) = (x'(t), y'(t))$ is tangent to the curve C at $s(t)$. Assume that s traverses C in the **counterclockwise** direction. Then $s'(t)$ will point outward and perpendicular to C if we rotate $s'(t)$ by $\pi/2$ radians in the clockwise direction. That is, $n(t) = (y'(t), -x'(t))$ points normal to C in the outward direction. We therefore have $e_n(t) = n(t) / \|n(t)\| = (y'(t), -x'(t)) / \sqrt{(x'(t))^2 + (y'(t))^2}$, and

$$\begin{aligned} \int_C F \cdot e_n ds &= \int_{t=a}^{t=b} (F_1(x(t), y(t)), F_2(x(t), y(t))) \cdot \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= \int_{t=a}^{t=b} F_1(x(t), y(t)) \frac{d}{dt} y(t) - F_2(x(t), y(t)) \frac{d}{dt} x(t) dt = \int_{t=a}^{t=b} (F(s(t))) \cdot n(t) dt. \end{aligned}$$

To emphasize that $\int_C F \cdot e_n ds$ is the integral of a closed curve, we sometimes add a closed circle to the integral, as follows.

$$\int_C F \cdot e_n ds = \oint_C F \cdot e_n ds.$$

Example 5.17. Let $F(x, y) = (x, y)$ be a vector field, and let $s(t) = (\cos t, \sin t)$ be a parametrization of the circle C , where $t \in [0, 2\pi]$. Note that $s'(t) = (-\sin t, \cos t)$, so that $\|s'(t)\| = 1$, and $n(t) = (\cos t, \sin t)$. Also, s moves counterclockwise. So, we have

$$\begin{aligned} \oint_C F \cdot e_n ds &= \int_{t=0}^{t=2\pi} F(s(t)) \cdot (\cos t, \sin t) dt = \int_{t=0}^{t=2\pi} (\cos t, \sin t) \cdot (\cos t, \sin t) dt \\ &= \int_{t=0}^{t=2\pi} (\cos^2 t + \sin^2 t) dt = 2\pi. \end{aligned}$$

Example 5.18. Let's compute the flux of $F(x, y) = (x, y)$ moving to the left across the parametrized straight line $s(t) = (1, t^2)$ where $0 \leq t \leq 1$. Then $s'(t) = (0, 2t)$, so $n(t) = (-2t, 0)$ is normal to the curve and pointing to the left. That is,

$$\int_C F \cdot e_n ds = \int_{t=0}^{t=1} F(s(t)) \cdot n(t) dt = \int_{t=0}^{t=1} (1, t^2) \cdot (-2t, 0) dt = \int_{t=0}^{t=1} (-2t) dt = -1.$$

If we computed the flux moving to the right across the line s , we would use $n(t) = (2t, 0)$ and get $+1$ for the flux moving to the right. Note also that if we reverse the direction of the parametrization, e.g. using $r(t) = (1, t - 1)$, where $0 \leq t \leq 1$, then we still get the same answer. In this case, $r'(t) = (0, 1)$, so $n(t) = (-1, 0)$ points to the left, and

$$\int_C F \cdot e_n dr = \int_{t=0}^{t=1} F(r(t)) \cdot n(t) dt = \int_{t=0}^{t=1} (1, t - 1) \cdot (-1, 0) dt = \int_{t=0}^{t=1} (-1) dt = -1.$$

5.2. Conservative Vector Fields. Path integrals can become quite difficult to compute if the path is very complicated. Luckily, many vector fields have a property where the integral along a path only depends on the endpoints of the path. We call these vector fields conservative.

Definition 5.19 (Conservative Vector Field). Let D be a domain in \mathbb{R}^3 . Let $F: D \rightarrow \mathbb{R}^3$ be a continuous vector field. Let C_1, C_2 be continuously differentiable paths in \mathbb{R}^3 with the same beginning point P and the same ending point Q . Assume that

$$\int_{C_1} F \cdot T ds = \int_{C_2} F \cdot T ds.$$

We then say that the path integral of F is **path independent** from P to Q . If for every P, Q in D , the path integral of F is path independent from P to Q , we then say that F is a **conservative vector field**.

Example 5.20. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real valued function on Euclidean space \mathbb{R}^3 . Consider the gradient vector field $\nabla f(x, y, z)$. We will show that $\nabla f(x, y, z)$ is a conservative vector field. Let $s(t) = (x(t), y(t), z(t))$ be a continuously differentiable parameterized path in Euclidean space \mathbb{R}^3 , so that $s: [0, 1] \rightarrow \mathbb{R}^3$. Let $g(t) = f(x(t), y(t), z(t))$. From the Fundamental Theorem of Calculus,

$$\int_{t=0}^{t=1} g'(t) dt = g(1) - g(0).$$

From the Chain rule,

$$g'(t) = \frac{\partial f}{\partial x} \frac{d}{dt} x(t) + \frac{\partial f}{\partial y} \frac{d}{dt} y(t) + \frac{\partial f}{\partial z} \frac{d}{dt} z(t) = (\nabla f)(x(t), y(t), z(t)) \cdot (s'(t)).$$

Therefore,

$$\int_C \nabla f \cdot T ds = \int_{t=0}^{t=1} (\nabla f)(s(t)) \cdot s'(t) dt = \int_{t=0}^{t=1} g'(t) dt = g(1) - g(0) = f(s(1)) - f(s(0)). \quad (*)$$

That is, $\int_C \nabla f \cdot T ds$ depends only on the endpoints $s(1)$ and $s(0)$ of the path s . Therefore, ∇f is a conservative vector field.

Remark 5.21. Equation $(*)$ can be considered a slight generalization of the fundamental theorem of calculus.

Example 5.22. Returning to Example 5.7, we know that the vector field $F(x, y, z) = (x, y, z)$ is conservative, since $F = \nabla f$, where $f(x, y, z) = (1/2)(x^2 + y^2 + z^2)$.

Below is an equivalent characterization of conservative vector fields.

Theorem 5.23 (Closed Loop Characterization of Conservative Vector Fields). *Let F be a vector field defined on a domain D . Then the following statements are equivalent.*

- F is a conservative vector field on D . (For any two paths C_1, C_2 in D with the same endpoint, we have $\int_{C_1} F \cdot T ds = \int_{C_2} F \cdot T ds$.)
- For any closed loop C in D (i.e. if $s: [0, 1] \rightarrow D$ parameterizes C , then $s(0) = s(1)$), we have $\oint_C F \cdot T ds = 0$.

Proof. Assume that F is conservative. Let $s: [0, 1] \rightarrow \mathbb{R}^3$ be a closed loop, so that $s(0) = s(1)$. Let s_1 denote the path where $s_1(t) = s(t)$ for each $t \in [0, 1/2]$ and let s_2 denote the path where $s_2(t) = s(t)$ for each $t \in [1/2, 1]$. Since F is conservative, and since $s_1(0) = s(0) = s(1) = s_2(1)$ and $s_1(1/2) = s_2(1/2)$, we have

$$\int_{t=0}^{t=1/2} F(s_1(t)) \cdot s_1'(t) dt = \int_{t=1}^{t=1/2} F(s_2(t)) \cdot s_2'(t) dt.$$

That is,

$$\begin{aligned} 0 &= \int_{t=0}^{t=1/2} F(s_1(t)) \cdot s_1'(t) dt - \int_{t=1}^{t=1/2} F(s_2(t)) \cdot s_2'(t) dt \\ &= \int_{t=0}^{t=1/2} F(s_1(t)) \cdot s_1'(t) dt + \int_{t=1/2}^{t=1} F(s_2(t)) \cdot s_2'(t) dt = \int_{t=0}^{t=1} F(s(t)) \cdot s'(t) dt \end{aligned}$$

Now, assume that: for any parameterized curve $s: [0, 1] \rightarrow \mathbb{R}^3$ with $s(0) = s(1)$, we have $\int_{t=0}^{t=1} F \cdot s' dt = 0$. Let A, B be any two points in Euclidean space \mathbb{R}^3 . Let s_1 be a path with $s_1(0) = A$ and $s_1(1/2) = B$ and let s_2 be a path where $s_2(1) = A$ and $s_2(1/2) = B$. Consider the path s where $s(t) = s_1(t)$ for each $t \in [0, 1/2]$ and $s(t) = s_2(t)$ for each $t \in [1/2, 1]$. Then s is a closed loop, so

$$\int_{t=0}^{t=1} F(s(t)) \cdot s'(t) dt = 0.$$

However, we can split this integral in two parts as follows

$$\begin{aligned} \int_{t=0}^{t=1} F(s(t)) \cdot s'(t) dt &= \int_{t=0}^{t=1/2} F(s_1(t)) \cdot s_1'(t) dt + \int_{t=1/2}^{t=1} F(s_2(t)) \cdot s_2'(t) dt \\ &= \int_{t=0}^{t=1/2} F(s_1(t)) \cdot s_1'(t) dt - \int_{t=1}^{t=1/2} F(s_2(t)) \cdot s_2'(t) dt \end{aligned}$$

So, $\int_{t=0}^{t=1/2} F(s_1(t)) \cdot s_1'(t) dt = \int_{t=1}^{t=1/2} F(s_2(t)) \cdot s_2'(t) dt$, as desired. \square

Definition 5.24. We say that a domain D is **connected** if any two points in D can be joined by a path which lies entirely in D . For example, \mathbb{R}^2 and \mathbb{R}^3 are connected, but \mathbb{R}^2 with an annulus removed is not connected.

Theorem 5.25 (Existence of Potential Function). *Let F be a conservative vector field on an open connected domain D . Then, there exists a function f such that $F = \nabla f$.*

Proof Sketch. Fix a point P in D . For any Q in D , define $f(Q)$ as any path integral of $F \cdot T$ from P to Q . Since F is conservative, this definition does not depend on the choice of path integral from P to Q . The Fundamental Theorem of Calculus implies that $\nabla f = F$. \square

Remark 5.26. This Theorem says that the function f exists, but it may not be easy to calculate f explicitly. For technical reasons, it is easier to find f when the domain D is **simply connected**. A domain D is simply connected if and only if it is connected, and every simple closed curve in D can be shrunk to a point entirely within D . For example, \mathbb{R}^2 and \mathbb{R}^3 are simply connected. However, \mathbb{R}^2 with a disc removed is not simply connected, since a loop that encloses the removed disc cannot be shrunk to a point. Yet, being simply connected is **not** synonymous with having “no holes.” For example, \mathbb{R}^3 with a ball removed is simply connected. We will not emphasize the subject of simple connectedness very much in this course, though it turns out to be intimately connected with the topic of conservative vector fields.

The Closed Loop characterization is nice to know, but it still does not give us a nice way of checking whether or not a vector field is conservative. Luckily, we can test whether or not a vector field is conservative using the following facts.

Proposition 5.27 (Cross-Partials Test for Conservative Vector Fields on Simply Connected Domains). *Let D be a simply connected domain in \mathbb{R}^3 . Let $F: D \rightarrow \mathbb{R}^3$ be a continuously differentiable vector field. We write*

$$F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)).$$

Assume that the following equalities hold:

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Then F is conservative. So, there exists $f: D \rightarrow \mathbb{R}$ with $F = \nabla f$.

Similarly, if D is a simply connected domain in \mathbb{R}^2 , and $F: D \rightarrow \mathbb{R}^2$ is continuously differentiable with $F(x, y) = (F_1(x, y), F_2(x, y))$, and if $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, then F is conservative. So, there exists $f: D \rightarrow \mathbb{R}$ with $F = \nabla f$.

Remark 5.28. Suppose $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ has two continuous derivatives. Let’s verify that ∇f satisfies the cross-partial test. Recall, we already know that ∇f is a conservative vector field. Then $F = (F_1, F_2, F_3) = \nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$. Then the Component Test reduces to the fact that mixed partial derivatives commute. For example,

$$\frac{\partial}{\partial y} F_3 = \frac{\partial}{\partial y} \partial f/\partial z = \frac{\partial}{\partial z} \partial f/\partial y = \frac{\partial}{\partial z} F_2$$

Example 5.29. Consider the vector field $F(x, y, z) = (yz, xz, xy)$. In this case, perhaps it is possible to guess that $f(x, y, z) = xyz$ satisfies $F = \nabla f$, so F is conservative.

Example 5.30. Consider the vector field $F(x, y, z) = (F_1, F_2, F_3) = (2x - 3, -z, \cos(z))$. Observe that

$$\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}(\cos(z)) = 0, \quad \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}(-z) = -1.$$

So, from the Component Test, we see that F is not conservative.

Remark 5.31. If F is a conservative vector field, then $F = \nabla f$ for some function f . So, if F is conservative, then F passes the cross-partial test. However, *the converse is false in general*. If F satisfies the cross-partial test, then F is not necessarily conservative. If F satisfies the cross-partial test, then in general, F is conservative only if F is defined on a simply-connected domain.

Example 5.32. Consider the vector field $F(x, y) = (-y/(x^2 + y^2), x/(x^2 + y^2))$. Then F satisfies the cross-partial test $\partial F_2/\partial x - \partial F_1/\partial y = 0$, but F is not actually conservative, which we can see by integrating f on the unit circle. Let $s(t) = (\cos t, \sin t)$ where $0 \leq t < 2\pi$. Then $s'(t) = (-\sin t, \cos t)$, so

$$\begin{aligned} \oint_C F \cdot T ds &= \int_{t=0}^{t=2\pi} F(\cos t, \sin t) \cdot s'(t) dt = \int_{t=0}^{t=2\pi} F(\cos t, \sin t) \cdot (-\sin t, \cos t) dt \\ &= \int_{t=0}^{t=2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_{t=0}^{t=2\pi} dt = 2\pi \neq 0. \end{aligned}$$

Since F has a nonzero line integral on a closed loop, we conclude that F is not conservative. Note that F is undefined at $(0, 0)$, so F is defined on the plane with the origin removed, which is not a simply connected domain. So, the cross-partials test does *not* imply that F is conservative, in this case.

Example 5.33. Consider the vector field $F(x, y, z) = (e^x \cos y + yz, xz - e^x \sin y, xy + z)$. The cross-partial test shows that F is conservative. But how can we find a function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\nabla f = F$? It is perhaps not as easy to guess in this case. To find f , we will integrate F . Since f satisfies $\nabla f = F$, we know that f satisfies

$$\frac{\partial}{\partial x} f = e^x \cos y + yz, \quad \frac{\partial}{\partial y} f = xz - e^x \sin y, \quad \frac{\partial}{\partial z} f = xy + z. \quad (*)$$

By using some integrations and differentiations, this system of equalities will eventually tell us what f is. We use an indefinite integral with respect to x for the first equality while holding y, z fixed, and we apply the Fundamental Theorem of Calculus. We conclude that there exists a function $g(y, z)$ such that

$$f(x, y, z) = e^x \cos y + xyz + g(y, z). \quad (**)$$

Taking the y derivative of this equality, we get

$$\frac{\partial}{\partial y} f = -e^x \sin y + xz + \frac{\partial}{\partial y} g.$$

Comparing this to $(*)$, we see that g satisfies $\partial g/\partial y = 0$. That is, g does not depend on y . Therefore, there exists a function $h(z)$ such that $(**)$ becomes

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

Taking the z derivative of this equality, we get

$$\frac{\partial}{\partial z} f = xy + h'(z).$$

Comparing this to $(*)$, we see that $h'(z) = z$. Integrating this, there exists a constant c such that $h(z) = (1/2)z^2 + c$. In conclusion,

$$f(x, y, z) = e^x \cos y + xyz + (1/2)z^2 + c.$$

5.3. Parametric Surfaces. We are used to describing the unit circle in two different ways. On the one hand, the unit circle is the set of points (x, y) in the plane \mathbb{R}^2 such that $x^2 + y^2 = 1$. On the other hand, we can describe the unit circle using the parametrized path $s(t) = (\cos t, \sin t)$ where $0 \leq t < 2\pi$ and $s: [0, 2\pi) \rightarrow \mathbb{R}^2$. We could also parametrize the circle using $s(t) = (\cos t^2, \sin t^2)$ where $0 \leq t < \sqrt{2\pi}$, or using $s(t) = (\cos t, \sin t)$ where $-2\pi \leq t < 0$. That is, we make a distinction between the circle itself (which is a set of points), and a parametrization of the surface (which is a function that maps onto the circle). Also, we note that different parametrizations can describe the same curve, and the direction of motion (i.e. orientation) of different parametrizations can be different.

We similarly think of surfaces as sets of points, and also by their parametrizations. And these parametrizations have orientations, which affect our integration theory. In your previous course, we discussed quadric surfaces. For example, the unit sphere is described as the set of all (x, y, z) in Euclidean space \mathbb{R}^3 such that $x^2 + y^2 + z^2 = 1$. For the purpose of integration, we need to parametrize this surface. We can describe the sphere using spherical coordinates. That is, we can use the parametrization

$$G(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi), \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi.$$

For parametrized paths s , we think of s as a map where the domain of s is an interval. For parametrized surfaces G , we think of G as a map where its domain is a region D in the plane. In the case of the above parametrization of the sphere, we have $G: D \rightarrow \mathbb{R}^3$, where $D = \{(\phi, \theta): 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi\}$. That is, G maps the rectangle D onto the unit sphere.

We could also use a different parametrization, using cylindrical coordinates. For example, consider

$$G(z, \theta) = (\sqrt{1 - z^2} \cos \theta, \sqrt{1 - z^2} \sin \theta, z), \quad 0 \leq \theta < 2\pi, \quad -1 \leq z \leq 1.$$

Then G also parametrizes the unit sphere where $x^2 + y^2 = 1 - z^2$.

Example 5.34. Suppose we have a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. We think of the function f as defining a surface $z = f(x, y)$. This surface is known as the graph of the function f . We can realize the surface $z = f(x, y)$ as a parametrized surface by the formula

$$G(x, y) = (x, y, f(x, y)).$$

Example 5.35. The cone $z = \sqrt{x^2 + y^2}$, $z \geq 0$ can be parametrized by the function $G(x, y) = (x, y, \sqrt{x^2 + y^2})$, where $x, y \in \mathbb{R}$

Example 5.36. The double cone $z^2 = x^2 + y^2$ can be parametrized by

$$G(z, \theta) = (z \cos \theta, z \sin \theta, z), \quad z \in \mathbb{R}, \theta \in [0, 2\pi).$$

In particular, the cone $z = \sqrt{x^2 + y^2}$ can also be parametrized in this way, where $z \geq 0$ and $\theta \in [0, 2\pi)$.

Example 5.37. The cylinder $x^2 + y^2 = 1$ with $0 \leq z \leq 1$ can be parametrized by

$$G(z, \theta) = (\cos \theta, \sin \theta, z), \quad 0 \leq z \leq 1, \theta \in [0, 2\pi).$$

5.4. Surface Integrals. Suppose we have a line $y = ax$ in the plane \mathbb{R}^2 . Over the x interval where $x \in [0, 1]$, the line $y = ax$ has length

$$\sqrt{1 + a^2} = \sqrt{1 + (dy/dx)^2}.$$

This follows from the Pythagorean Theorem.

Since any continuously differentiable function is locally linear, we therefore define the **arc length** of a single variable function $y = f(x)$ from $x = a$ to $x = b$ as

$$\int_{x=a}^{x=b} \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx.$$

With this definition in mind, we now consider how to compute the surface area of a parameterized surface.

Consider a plane $z = ax + by$ where a, b are constants and (x, y, z) is a variable point in Euclidean space \mathbb{R}^3 .

The plane satisfies $f(x, y, z) = ax + by - z = 0$, so that $\nabla f(x, y, z) = (a, b, -1)$. So, the plane $z = ax + by$ has normal vector $(a, b, -1)$, so a unit normal vector e_n is

$$e_n = \frac{(a, b, -1)}{\sqrt{1 + a^2 + b^2}}$$

Consider the part of the plane sitting above the square where $0 \leq x, y \leq 1$. The square has area 1, but what is the area of the plane above this square? The plane above the square is a parallelogram with vertices $(0, 0, 0)$, $(0, 1, b)$, $(1, 0, a)$ and $(1, 1, a + b)$. So, the area of the parallelogram is

$$\|(0, 1, b) \times (1, 0, a)\| = \|(a, b, -1)\| = \sqrt{1 + a^2 + b^2}. \quad (1)$$

In conclusion, the area of the plane $z = ax + by$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is

$$\sqrt{1 + a^2 + b^2} = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Since a continuously differentiable surface looks locally like its tangent plane, we therefore define surface area as follows.

Definition 5.38 (Surface Area). Let $z = f(x, y)$ be the equation for a surface S , where (x, y, z) is a point in Euclidean space \mathbb{R}^3 . Let D be a region in the plane \mathbb{R}^2 . We define the **surface area** of the surface S over the region D by

$$\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy.$$

We can also interpret the plane $z = ax + by$ as a parameterized surface. The plane is then given by the set of points $(x, y, ax + by) = G(x, y)$ in Euclidean space \mathbb{R}^3 , where (x, y) is a variable point in the plane \mathbb{R}^2 . In this case, we write the area of the parallelogram (1) as

$$\|(0, 1, b) \times (1, 0, a)\| = \|(\partial G/\partial y) \times (\partial G/\partial x)\| = \|(\partial G/\partial x) \times (\partial G/\partial y)\|.$$

We therefore define the surface area of a parameterized surface as follows.

Definition 5.39 (Surface Area). Let D be a region in the plane \mathbb{R}^2 . Let $G(v, w) = (x(v, w), y(v, w), z(v, w))$ be a parametrization of a surface S . That is, $G: D \rightarrow \mathbb{R}^3$, so that, where $(x(v, w), y(v, w), z(v, w))$ is a point in Euclidean space \mathbb{R}^3 , for each (v, w) in D . We define the **surface area** of the S over the region D by

$$\iint_D \|(\partial G/\partial v) \times (\partial G/\partial w)\| \, dv dw.$$

As in the case of arc length, we need to assume that G does not overlap onto itself, otherwise we might double-count some parts of the surface. We therefore assume that $G(v, w) \neq G(a, b)$ whenever $(v, w) \neq (a, b)$.

Example 5.40. Let's find the surface area of the two-dimensional sphere. The sphere S of radius 1 is defined as the set of points (x, y, z) in Euclidean space \mathbb{R}^3 where $x^2 + y^2 + z^2 = 1$. Using spherical coordinates, we can parametrize S using the map $G(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$. Here $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$. The surface area of the sphere is then given by

$$\begin{aligned} & \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \|(\partial G/\partial \phi) \times (\partial G/\partial \theta)\| \, d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \|(\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi) \times (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)\| \, d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \|(\sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi \cos^2 \theta + \sin \phi \cos \phi \sin^2 \theta)\| \, d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \|(\sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi)\| \, d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \theta} \, d\theta d\phi = \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \sqrt{\sin^2 \phi} \, d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \sin \phi \, d\theta d\phi \quad , \text{ since } \sin \phi \geq 0 \text{ for } \phi \in [0, \pi] \\ &= \int_{\phi=0}^{\phi=\pi} 2\pi \sin \phi \, d\phi = 2\pi[-\cos \phi]_{\phi=0}^{\phi=\pi} = 4\pi. \end{aligned}$$

More generally, the sphere of radius r has surface area $4\pi r^2$, which can be seen by doing the same computation for $G(\phi, \theta) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$.

Definition 5.41 (Parameterized Surface Integral of a Function). Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real valued function on Euclidean space \mathbb{R}^3 . Let D be a region in the plane \mathbb{R}^2 . Let $G(v, w) = (x(v, w), y(v, w), z(v, w))$ be a parametrization of a surface S . That is, $G: D \rightarrow \mathbb{R}^3$, so that $(x(v, w), y(v, w), z(v, w))$ is a point in Euclidean space \mathbb{R}^3 , for each (v, w) in the plane \mathbb{R}^2 . Assume $G(v, w) \neq G(a, b)$ whenever $(v, w) \neq (a, b)$. We define the **surface integral** of f on the surface S over the region D by

$$\iint_S f \, dS = \iint_D f(G(v, w)) \|(\partial G/\partial v) \times (\partial G/\partial w)\| \, dv dw.$$

Example 5.42. Let's integrate the function $f(x, y, z) = 2(x^2 + y^2)$ over the cylinder S where $x^2 + y^2 = 1$ and $0 \leq z \leq 1$. We first parametrize the cylinder by $G(z, \theta) = (\cos \theta, \sin \theta, z)$ where D consists of all (z, θ) where $0 \leq \theta < 2\pi$ and $0 \leq z \leq 1$. We then have

$$\begin{aligned} \iint_S f dS &= \iint_D f(G(z, \theta)) \|(\partial G/\partial z) \times (\partial G/\partial \theta)\| d\theta dz \\ &= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} f(\cos \theta, \sin \theta, z) \|(0, 0, 1) \times (-\sin \theta, \cos \theta, 0)\| d\theta dz \\ &= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} 2(\cos^2 \theta + \sin^2 \theta) \|(-\cos \theta, -\sin \theta, 0)\| d\theta dz \\ &= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} 2 d\theta dz = 4\pi. \end{aligned}$$

Remark 5.43. The change of variables formula in dimension two follows from our definition of surface integral as follows. Suppose the surface S is contained in the plane $z = 0$. Suppose also we can parameterize the surface S by some parametrization $G(x, y) = (a(x, y), b(x, y), 0)$.

$$\begin{aligned} \iint_S f dS &= \iint_D f(G(x, y)) \left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\| dx dy \\ &= \iint_D f(G(x, y)) \left\| \left(\frac{\partial a}{\partial x}, \frac{\partial b}{\partial x}, 0 \right) \times \left(\frac{\partial a}{\partial y}, \frac{\partial b}{\partial y}, 0 \right) \right\| dx dy \\ &= \iint_D f(G(x, y)) \left\| \left(0, 0, \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y} \right) \right\| dx dy = \iint_D f(G(x, y)) \left| \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y} \right| dx dy. \end{aligned}$$

The final integrand is then $f(G(x, y))$ multiplied by a two-dimensional Jacobian determinant.

Definition 5.44 (Parameterized Surface Integral of a Vector Field). Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field on Euclidean space \mathbb{R}^3 . Let D be a region in the plane \mathbb{R}^2 . Let $G(v, w) = (x(v, w), y(v, w), z(v, w))$ be a parametrization of a surface S . That is, $G: D \rightarrow \mathbb{R}^3$, so that, where $(x(v, w), y(v, w), z(v, w))$ is a point in Euclidean space \mathbb{R}^3 , for each (v, w) in the plane \mathbb{R}^2 . Assume $G(v, w) \neq G(a, b)$ whenever $(v, w) \neq (a, b)$. Let e_n denote a unit-length normal vector to the surface S . We define the **surface integral** of F on the surface S over the region D by

$$\begin{aligned} \iint_S F \cdot e_n dS &= \iint_D F(G(v, w)) \cdot \frac{(\partial G/\partial v) \times (\partial G/\partial w)}{\|(\partial G/\partial v) \times (\partial G/\partial w)\|} \|(\partial G/\partial v) \times (\partial G/\partial w)\| dv dw \\ &= \iint_D F(G(v, w)) \cdot ((\partial G/\partial v) \times (\partial G/\partial w)) dv dw. \end{aligned}$$

In the case that F is interpreted as the flow of a fluid, we call $\iint_S F \cdot e_n dS$ the **flux** of F across the surface S in the direction e_n .

Remark 5.45. If we interchange the roles of v and w , then the vector $(\partial G/\partial v) \times (\partial G/\partial w)$ changes sign. For this reason, a parameterized surface always has an implicit orientation. That is, when computing a surface integral, we will always specify the direction of the normal vector e_n . Some surfaces cannot be assigned an orientation (e.g. a Möbius strip), but we will not need to worry too much about this issue in this course.

Example 5.46. Find the flux of the vector field $F(x, y, z) = (x, y, z)$ outward through the cylinder S where $x^2 + y^2 = 1$ and $0 \leq z \leq 1$. As usual, we parameterize the cylinder using cylinder coordinates by $G(z, \theta) = (\cos \theta, \sin \theta, z)$ where the domain D is defined by $0 \leq \theta < 2\pi$ and $0 \leq z \leq 1$. We then compute

$$\begin{aligned} \iint_S F \cdot e_n dS &= \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} F(G(z, \theta)) \cdot \left(\left(\frac{\partial G}{\partial \theta} \right) \times \left(\frac{\partial G}{\partial z} \right) \right) dz d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} (\cos \theta, \sin \theta, z) \cdot ((-\sin \theta, \cos \theta, 0) \times (0, 0, 1)) dz d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} (\cos \theta, \sin \theta, z) \cdot (\cos \theta, \sin \theta, 0) dz d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} (\cos^2 \theta + \sin^2 \theta) dz d\theta = 2\pi. \end{aligned}$$

Note that the normal vector $(\partial G/\partial \theta) \times (\partial G/\partial z)$ is actually outward pointing, so we have selected the correct normal vector. For example, if $\theta = 0$, then $(x, y, z) = (1, 0, 0)$ and $(\partial G/\partial \theta) \times (\partial G/\partial z) = (1, 0, 0)$.

Example 5.47. Find the flux of the field $F(x, y, z) = (yz, x, -z^2)$ in the direction pointing outward of the parabolic cylinder $y = x^2$, where $0 \leq x \leq 1$ and $0 \leq z \leq 4$.

We parameterize the parabolic cylinder S using the x and z coordinates. That is, we define $G(x, z) = (x, x^2, z)$. We then compute

$$\begin{aligned} \iint_S F \cdot e_n dS &= \int_{x=0}^{x=1} \int_{z=0}^{z=4} F(G(x, z)) \cdot ((\partial G/\partial x) \times (\partial G/\partial z)) dz dx \\ &= \int_{x=0}^{x=1} \int_{z=0}^{z=4} F(x, x^2, z) \cdot ((1, 2x, 0) \times (0, 0, 1)) dz dx \\ &= \int_{x=0}^{x=1} \int_{z=0}^{z=4} (x^2 z, x, -z^2) \cdot (2x, -1, 0) dz dx = \int_{x=0}^{x=1} \int_{z=0}^{z=4} (2x^3 z - x) dz dx \\ &= \int_{x=0}^{x=1} [x^3 z^2 - xz]_{z=0}^{z=4} dx = \int_{x=0}^{x=1} (16x^3 - 4x) dx = [4x^4 - 2x^2]_{x=0}^{x=1} = 2. \end{aligned}$$

Note that the normal vector $(2x, -1, 0)$ always has negative y -component, so the normal is pointing outward.

6. GREEN'S THEOREM

The fundamental theorem of calculus is an extremely useful tool in single variable calculus. We therefore look for analogues of the fundamental theorem in two and three dimensions.

Recall that, for a continuously differentiable function $g: [a, b] \rightarrow \mathbb{R}$, we have

$$g(b) - g(a) = \int_a^b g'(t) dt.$$

Specifically, the integral of g' on the interval $[a, b]$ reduces to evaluating g on the boundary of the interval $[a, b]$.

We will now begin to formulate analogues of this statement for double and triple integrals. The main idea will be the same. In some cases, the integral over a region D in the plane will reduce to computing a line integral on the boundary of D . And in some cases, the

integral over a region D in Euclidean space will reduce to computing a surface integral on the boundary of D . We begin our investigation in the plane.

Definition 6.1 (Curl, Dimension 2). Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a differentiable vector field. We write $F = (F_1, F_2)$. Define the **curl** of F for any (x, y) in the plane by

$$\text{curl}(F)(x, y) = \frac{\partial}{\partial x} F_2(x, y) - \frac{\partial}{\partial y} F_1(x, y).$$

Remark 6.2. If $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a conservative vector field, then $\text{curl}(F) = 0$. Conversely, if $\text{curl}(F) = 0$, and if F is defined on a simply-connected domain, then F is conservative. So, the curl of F measures how far away F is from being conservative.

Remark 6.3. We can also interpret the curl as how much the vector field F is “rotating.” For example, recall the vector field $F(x, y) = (-y, x)$, which looks like a whirlpool. Note that $\text{curl}(F)(x, y) = 1 - (-1) = 2$ for all (x, y) in the plane \mathbb{R}^2 .

Example 6.4. Recall that the vector field $F(x, y) = (x, y)$ is always pointing away from the origin. And $\text{curl}(F)(x, y) = 0$.

Theorem 6.5 (Green’s Theorem, Dimension 2). Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuously differentiable vector field. We write $F = (F_1, F_2)$. Let $s: [a, b] \rightarrow \mathbb{R}^2$ be a parametrization of a simple closed curve C in the plane \mathbb{R}^2 , so that s moves in the counterclockwise direction. (So, $s(a) = s(b)$.) Let D be the region in the plane bounded by s . Then

$$\begin{aligned} \int_C F \cdot T ds &= \int_{t=a}^{t=b} F(s(t)) \cdot s'(t) dt = \iint_D \text{curl}(F) dA \\ &= \iint_D \left(\frac{\partial}{\partial x} F_2(x, y) - \frac{\partial}{\partial y} F_1(x, y) \right) dx dy. \end{aligned}$$

The left side is sometimes written as $\oint_C F \cdot T ds$ or $\oint_C F_1 dx + F_2 dy$.

Remark 6.6. The left side of Green’s Theorem is a one-variable integral over the boundary of D , while the right side is a two-variable integral over D . Also, the right side involves derivatives of F , whereas the left side only involves the values of F . For these reasons, Green’s Theorem can be considered a generalization of the single-variable Fundamental Theorem of calculus.

Corollary 6.7. Let D be a simply connected region in the plane. Let $F: D \rightarrow \mathbb{R}^2$ satisfy the cross-partial test (so that $\text{curl}(F) = 0$). Then the integral of F along any simple closed curve is zero, so F is conservative, i.e. we proved that the cross-partial test works. To see this, let C be any closed curve in D , and then let \tilde{D} be a region in D whose boundary is C .

Remark 6.8. For any real-valued function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ (with two continuous derivatives), recall that the partial derivatives of f commute, so

$$\text{curl}(\nabla f) = 0.$$

Example 6.9. Let’s try to see why Green’s Theorem is true. Consider the rectangle $R = [a, b] \times [c, d]$ in the plane. Let C be the boundary of the rectangle. Moving in the

counterclockwise direction, this rectangle has vertices (b, c) , (b, d) , (a, d) and (a, c) . Using the single-variable Fundamental Theorem of Calculus, we have

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} \frac{\partial}{\partial x} F_2(x, y) dx dy = \int_{y=c}^{y=d} (F_2(b, y) - F_2(a, y)) dy.$$

Similarly,

$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} \frac{\partial}{\partial y} F_1(x, y) dy dx = \int_{x=a}^{x=b} (F_1(x, d) - F_1(x, c)) dx.$$

So, subtracting these two things,

$$\begin{aligned} & \int_{y=c}^{y=d} \int_{x=a}^{x=b} \frac{\partial}{\partial x} F_2(x, y) dx dy - \int_{x=a}^{x=b} \int_{y=c}^{y=d} \frac{\partial}{\partial y} F_1(x, y) dy dx \\ &= \int_{y=c}^{y=d} F_2(b, y) dy + \int_{x=b}^{x=a} F_1(x, d) dx + \int_{y=d}^{y=c} F_2(a, y) dy + \int_{x=a}^{x=b} F_1(x, c) dx. \end{aligned}$$

The first term is the term on the right side of Green's Theorem. The second term is the term on the left side of Green's Theorem. To see this, suppose s travels at unit speed in four straight lines, traversing the vertices (b, c) , (b, d) , (a, d) and (a, c) in counterclockwise order. While s goes from (b, c) to (b, d) , we have $s(t) = (b, t)$, $c \leq t \leq d$. So, $s'(t) = (0, 1)$, and $F(s(t)) \cdot s'(t) = F_2(s(t)) = F_2(b, t)$, so

$$\int_{t=c}^{t=d} F(s(t)) \cdot s'(t) dt = \int_{t=c}^{t=d} F_2(b, t) dt = \int_{y=c}^{y=d} F_2(b, y) dy.$$

While s goes from (b, d) to (a, d) , we have $s(t) = (a + b - t, d)$, $a \leq t \leq b$. So, $s'(t) = (-1, 0)$, and $F(s(t)) \cdot s'(t) = -F_1(s(t)) = -F_1(a + b - t, d)$, so

$$\int_{t=a}^{t=b} F(s(t)) \cdot s'(t) dt = \int_{t=a}^{t=b} -F_1(a + b - t, d) dt = \int_{x=b}^{x=a} F_1(x, d) dx.$$

Here we used the change of variables $x = a + b - t$.

While s goes from (a, d) to (a, c) , we have $s(t) = (a, c + d - t)$, $c \leq t \leq d$. So, $s'(t) = (0, -1)$, and $F(s(t)) \cdot s'(t) = -F_2(s(t)) = -F_2(a, c + d - t)$, so

$$\int_{t=c}^{t=d} F(s(t)) \cdot s'(t) dt = \int_{t=c}^{t=d} -F_2(a, c + d - t) dt = \int_{y=d}^{y=c} F_2(a, y) dy.$$

Here we used the change of variables $y = c + d - t$.

While s goes from (a, c) to (b, c) , we have $s(t) = (t, c)$, $a \leq t \leq b$. So, $s'(t) = (1, 0)$, and $F(s(t)) \cdot s'(t) = F_1(s(t)) = F_1(t, c)$, so

$$\int_{t=a}^{t=b} F(s(t)) \cdot s'(t) dt = \int_{t=a}^{t=b} F_1(t, c) dt = \int_{x=a}^{x=b} F_1(x, c) dx.$$

Adding up all four terms, we conclude that

$$\oint_C F \cdot T ds = \int_{y=c}^{y=d} F_2(b, y) dy + \int_{x=b}^{x=a} F_1(x, d) dx + \int_{y=d}^{y=c} F_2(a, y) dy + \int_{x=a}^{x=b} F_1(x, c) dx.$$

In conclusion, we have verified Green's Theorem for rectangles D . One can then subdivide a general region D into a grid of small rectangles, and verify that when two adjacent rectangles touch, their touching "boundary terms" cancel each other. See Figure 1.

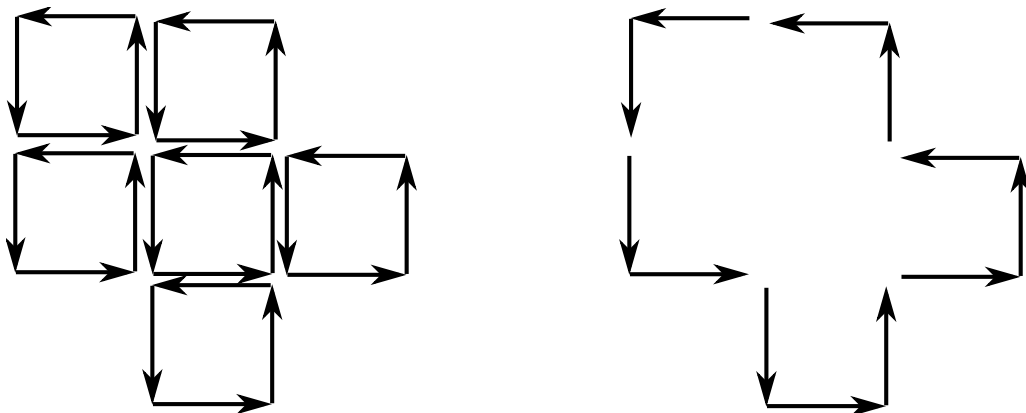


FIGURE 1. Depiction of a Proof of Green's Theorem

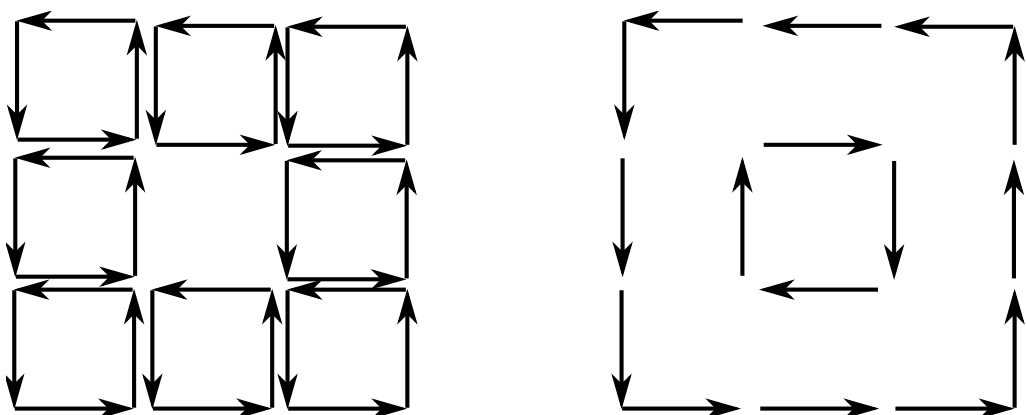


FIGURE 2. Depiction of a Proof of Generalized Green's Theorem

The above argument actually proves the following more general result.

Theorem 6.10 (Generalized Green's Theorem, Dimension 2). *Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuously differentiable vector field. Let D be a region in the plane whose boundary C consists of a finite number of simple closed curves. Suppose C is the union of simple closed curves C_1, \dots, C_k . For each i with $1 \leq i \leq k$, we orient C_i so that, when we travel forward along C_i , the region D lies on the left side of C_i . Then*

$$\oint_C F \cdot T \, ds = \iint_D \text{curl}(F) \, dA.$$

Example 6.11. Let $D = [0, 1] \times [0, 1]$ and let $F(x, y) = (y^2, x + y)$. We will compute both sides of Green's Theorem. Note that $\text{curl}(F)(x, y) = 1 - 2y$. So,

$$\iint_D \text{curl}(F)(x, y) \, dx \, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=1} (1 - 2y) \, dy \, dx = 0.$$

Now, let's compute the integral of $F \cdot T$ on the boundary of D . Note that the boundary C of D consists of the four edges of the square D . We use four separate line segments, traversing the square D in the counterclockwise direction. Let $s_1(t) = (t, 0)$, $s_2(t) = (1, t)$,

$s_3(t) = (1 - t, 1)$, $s_4(t) = (0, 1 - t)$, where $0 \leq t \leq 1$. Then

$$\begin{aligned} \oint_C F \cdot T ds &= \int_{t=0}^{t=1} F(s_1(t)) \cdot s_1'(t) dt + \int_{t=0}^{t=1} F(s_2(t)) \cdot s_2'(t) dt \\ &\quad + \int_{t=0}^{t=1} F(s_3(t)) \cdot s_3'(t) dt + \int_{t=0}^{t=1} F(s_4(t)) \cdot s_4'(t) dt \\ &= \int_{t=0}^{t=1} (0, t) \cdot (1, 0) dt + \int_{t=0}^{t=1} (t^2, 1 + t) \cdot (0, 1) dt \\ &\quad + \int_{t=0}^{t=1} (1, 2 - t) \cdot (-1, 0) dt + \int_{t=0}^{t=1} ((1 - t)^2, 1 - t) \cdot (0, -1) dt \\ &= 0 + \int_{t=0}^{t=1} (1 + t) dt + \int_{t=0}^{t=1} (-1) dt + \int_{t=0}^{t=1} (t - 1) dt \\ &= 0 + (3/2) - 1 - (1/2) = 0. \end{aligned}$$

So, we have verified that $\oint_C F \cdot T ds = \iint_D \text{curl}(F) dA$.

Example 6.12. Let $F(x, y) = (x, x + y)$. Let D denote the region where $x^2 + y^2 \leq 4$. Then D is a disc with boundary circle C of radius 2. By Green's Theorem, if C is oriented counterclockwise, we get

$$\oint_C F \cdot T ds = \iint_D \text{curl}(F) dA = \iint_D dA = 4\pi.$$

Example 6.13. Let $F(x, y) = (x, x + y)$. Let D denote the region where $1 \leq x^2 + y^2 \leq 4$. Then D is an annulus with boundary C consisting of two circles. Suppose C_1 is the outer circle ($x^2 + y^2 = 4$) of the annulus, oriented counterclockwise, and C_2 is the inner circle ($x^2 + y^2 = 1$) of the annulus, oriented clockwise. Then by Green's Theorem,

$$\oint_C F \cdot T ds = \oint_{C_1} F \cdot T ds + \oint_{C_2} F \cdot T ds = \iint_D \text{curl}(F) dA = \iint_D dA = 4\pi - \pi = 3\pi.$$

7. STOKES' THEOREM

In order to state a version Green's Theorem in dimension 3 (which is known as Stokes' Theorem), we need to define the curl of a vector field in dimension 3. Unlike before, the curl of a 3-dimensional vector field will be a vector field, rather than a number.

Definition 7.1 (Curl, Dimension 3). Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. We write $F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$. Define the **curl** of F at (x, y, z) as the following vector in Euclidean space \mathbb{R}^3 .

$$\text{curl}(F)(x, y, z) = \left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2, \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3, \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \right).$$

The curl of F is sometimes written as $\nabla \times F$, since the following determinant "formula" helps to remember the definition of the curl.

$$\text{curl}(F) = \nabla \times F = \det \begin{pmatrix} (1, 0, 0) & (0, 1, 0) & (0, 0, 1) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Example 7.2. Let $F(x, y, z) = (y, z, -x^2)$. Then $\text{curl}(F)(x, y, z) = (-1, 2x, -1)$.

Theorem 7.3 (Stokes' Theorem). Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a continuously differentiable vector field. Let S be a surface in Euclidean space \mathbb{R}^3 . Let $s: [a, b] \rightarrow \mathbb{R}^3$ denote a parametrization of the boundary curve C of S . Let e_n denote a normal vector to the surface S . Then

$$\int_C F \cdot T ds = \iint_S \text{curl}(F) \cdot e_n dS.$$

The left side is often written as $\oint_C F \cdot T ds$. Implicit in this equality is that the normal vector e_n determines the orientation of s , via the right hand rule. That is, once the normal vector e_n is chosen, then s must rotate counterclockwise relative to e_n , via the right hand rule.

Remark 7.4. Strictly speaking, the surface S should be orientable, so that when the normal vector e_n moves along the surface, e_n always points on the same side of the surface. (One such surface that does not satisfy the condition is the Möbius strip.) However, this issue should not arise much in this class.

Remark 7.5. If D is contained in the plane $z = 0$, then Stokes' Theorem reduces to Green's Theorem. To see this, note that e_n becomes $e_n = (0, 0, 1)$, so $\text{curl}(F) \cdot e_n = \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1$ in this case, and then s has counterclockwise orientation.

Remark 7.6. Our justification for this theorem is similar to the justification for Green's Theorem.

Remark 7.7. Stokes' Theorem particularly implies that the surface integral $\iint_S \text{curl}(F) \cdot e_n dS$ does not depend on the parametrization G of the surface S , since the quantity on the left side of Stokes' Theorem does not depend on the parametrization G of the surface S . Moreover, no matter what surface S that appears on the right side, as long as it has boundary C , the quantity on the right remains unchanged.

Remark 7.8. If S has no boundary, e.g. if S is a sphere, then the left side of Stokes' Theorem is automatically zero.

Corollary 7.9. Let D be a simply connected region in \mathbb{R}^3 . Let $F: D \rightarrow \mathbb{R}^3$ satisfy the cross-partial test (so that $\text{curl}(F) = 0$). Then the integral of F along any simple closed curve is zero, so F is conservative, i.e. we proved that the cross-partial test works. To see this, let C be any closed curve in D , and then let \tilde{D} be a region in D whose boundary is C .

Remark 7.10. For any real-valued function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ (with two continuous derivatives), recall that the partial derivatives of f commute, so

$$\text{curl}(\nabla f) = 0.$$

Example 7.11. We verify that Stokes' Theorem holds when $F(x, y, z) = (y, -x, 0)$ and when G parametrizes the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, so that C is the circle $x^2 + y^2 = 1$, $z = 0$.

We first parametrize the surface S using Cartesian coordinates. Define

$$G(v, w) = (v, w, \sqrt{1 - v^2 - w^2}),$$

where $v^2 + w^2 \leq 1$. Then a normal vector n is given by

$$\begin{aligned} \frac{\partial}{\partial v}G \times \frac{\partial}{\partial w}G &= (1, 0, -v/\sqrt{1-v^2-w^2}) \times (0, 1, -w/\sqrt{1-v^2-w^2}) \\ &= (v/\sqrt{1-v^2-w^2}, w/\sqrt{1-v^2-w^2}, 1). \end{aligned}$$

Since n has positive z component, we see that n points above the hemisphere. We therefore must parameterize the circle C to rotate in the counterclockwise direction, in order to get equality in Stokes' Theorem (using the right hand rule). That is, we parameterize C by $s(t) = (\cos t, \sin t, 0)$, where $0 \leq t \leq 2\pi$.

Note that $\text{curl}(F) = (0, 0, -2)$. We now compute

$$\begin{aligned} \int_C F \cdot T ds &= \int_{t=0}^{t=2\pi} F(\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_{t=0}^{t=2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_{t=0}^{t=2\pi} (-\sin^2 t - \cos^2 t) dt = -2\pi. \end{aligned}$$

$$\begin{aligned} &\iint_S \text{curl}(F) \cdot e_n dS \\ &= \iint_{v^2+w^2 \leq 1} \text{curl}(F)(G(v, w)) \cdot (v/\sqrt{1-v^2-w^2}, w/\sqrt{1-v^2-w^2}, 1) \\ &= \iint_{v^2+w^2 \leq 1} (0, 0, -2) \cdot (v/\sqrt{1-v^2-w^2}, w/\sqrt{1-v^2-w^2}, 1) \\ &= \iint_{v^2+w^2 \leq 1} (-2) = -2\pi. \end{aligned}$$

Example 7.12. Consider the surface S which consists of all points where $x^2 + y^2 + z^2 = 1$ and $-1/2 \leq z \leq 1/2$. That is, S is a sphere with the top and bottom removed. Let C_1 denote the top circle $x^2 + y^2 = 3/4$ with $z = 1/2$, and let C_2 denote the bottom circle $x^2 + y^2 = 3/4$ with $z = -1/2$. Suppose e_n is the unit outward pointing normal vector to S . Then the right hand rule says that in Stokes' Theorem, we should orient the top circle C_1 in the clockwise direction, and we should orient the bottom circle C_2 in the counterclockwise direction. For example, we could parameterize C_1 by $s_1(t) = (\cos t, -\sin t, 1/2)$, and we could parameterize C_2 by $s_2(t) = (\cos t, \sin t, -1/2)$, where $0 \leq t < 2\pi$. Then Stokes' Theorem says

$$\oint_{C_1} F \cdot T ds_1 + \oint_{C_2} F \cdot T ds_2 = \iint_S \text{curl}(F) \cdot e_n dS.$$

Exercise 7.13. Consider the vector field $F(x, y, z) = (-y/(x^2 + y^2), x/(x^2 + y^2), z)$. Show that $\text{curl}(F) = 0$, but $\oint_C F \cdot ds$ is nonzero where C is the circle $x^2 + y^2 = 1$, $z = 0$. Explain why this does not contradict Stokes' Theorem.

8. DIVERGENCE THEOREM

We have seen that Stokes' Theorem reduces an integral over a surface to an integral over the boundary of the surface. And we saw that Green's Theorem is a special case of

Stokes' Theorem. We now make another statement to the same effect, though we now take derivatives in a bit of a different way. We also make a statement that says that some integral in a three-dimensional region reduces to computing an integral on a surface. Within these statements, instead of taking the curl of a vector field, we take the divergence. Whereas the curl is a vector, the divergence will always be a real number.

As before, we are still looking for generalizations of the single-variable Fundamental Theorem of Calculus.

Definition 8.1 (Divergence, Dimension 2). Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a differentiable vector field. We write $F(x, y) = (F_1(x, y), F_2(x, y))$. Define the **divergence** of F at the point (x, y) as the following real number.

$$\operatorname{div}(F)(x, y) = \frac{\partial}{\partial x} F_1(x, y) + \frac{\partial}{\partial y} F_2(x, y).$$

We sometimes write $\operatorname{div}(F) = \nabla \cdot F$.

Example 8.2. Let $F(x, y) = (x, y)$. Recall that F resembles a set of vectors emanating from the origin. Now, $\operatorname{div}(F)(x, y) = (\partial/\partial x)(x) + (\partial/\partial y)(y) = 2$. So, the divergence can be thought of as measuring how much the vector field is “spreading out” at each point. And $\operatorname{curl}(F)(x, y) = (\partial/\partial x)(y) - (\partial/\partial y)(x) = 0$, so this vector field does not “curl around” at all.

Example 8.3. Let $F(x, y) = (-y, x)$. Then F resembles a whirlpool and $\operatorname{curl}(F)(x, y) = (\partial/\partial x)(x) - (\partial/\partial y)(-y) = 2$. Now, $\operatorname{div}(F)(x, y) = (\partial/\partial x)(-y) + (\partial/\partial y)(x) = 0$. So, this whirlpool does not “spread out” at each point, since it is only “going around” clockwise.

Theorem 8.4 (Divergence Theorem, Dimension 2). Let D be a region in the plane \mathbb{R}^2 . Let C denote the boundary of D . Let e_n denote the exterior unit normal to C . Then

$$\iint_D \operatorname{div}(F) dA = \int_C F \cdot e_n ds$$

Remark 8.5. Note that the orientation of the boundary C does not matter, since the orientation does not change the value of $\int_C F \cdot e_n ds$. This situation contrasts sharply with Green's Theorem, where we had to pay close attention to the orientation of the boundary.

Remark 8.6. Recall that we had a generalized version of Green's Theorem in the plane that could deal with domains with holes. In the above statement of the Divergence Theorem, even if the domain D has holes, then the divergence theorem still holds as written. We simply always choose e_n to be the outward pointing unit normal vector.

Remark 8.7. Green's Theorem in Dimension 2 is equivalent to the Divergence Theorem in dimension 2. To see this, note that

$$\begin{aligned} \int_C (F_1, F_2) \cdot T ds &= \int_{t=a}^{t=b} (F_1(s(t)), F_2(s(t))) \cdot (x'(t), y'(t)) dt \\ &= \int_{t=a}^{t=b} (F_2(s(t)), -F_1(s(t))) \cdot (y'(t), -x'(t)) dt = \int_C (F_2, -F_1) \cdot e_n ds. \end{aligned}$$

And note that

$$\operatorname{curl}(F_1, F_2) = \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = \operatorname{div}(F_2, -F_1).$$

In summary, Green's Theorem for the vector field (F_1, F_2) is equivalent to the Divergence Theorem for the vector field $(F_2, -F_1)$. That is, Green's Theorem and the Divergence Theorem are equivalent.

Example 8.8. Compute the outward flux of the vector field $F(x, y) = (x^2 + 4y, x + y^2)$ out of the square C in the plane which is bounded by the lines $x = 0, x = 1$ and $y = 0, y = 1$.

From the Divergence Theorem, if we traverse C counterclockwise, then

$$\iint_D \operatorname{div}(F) dA = \oint_C F \cdot e_n ds$$

So, to compute the desired flux, it suffices to compute $\iint_D \operatorname{div}(F) dA$. We therefore compute

$$\begin{aligned} \iint_D \operatorname{div}(F) dA &= \int_{y=0}^{y=1} \int_{x=0}^{x=1} \operatorname{div} F(x, y) dx dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} (2x + 2y) dx dy \\ &= \left(\int_{x=0}^{x=1} 2x dx \right) + \left(\int_{y=0}^{y=1} 2y dy \right) = 1 + 1 = 2. \end{aligned}$$

Note that computing the line integral would involve more effort in this case.

Example 8.9. Let $F(x, y) = (x, x + y)$. Let D denote the region where $1 \leq x^2 + y^2 \leq 4$. Then D is an annulus with boundary C consisting of two circles. Suppose C_1 is the outer circle ($x^2 + y^2 = 4$) of the annulus, and C_2 is the inner circle ($x^2 + y^2 = 1$) of the annulus. Let e_n be the exterior unit normal to D . Then by the Divergence Theorem,

$$\oint_C F \cdot e_n ds = \oint_{C_1} F \cdot e_n ds + \oint_{C_2} F \cdot e_n ds = \iint_D \operatorname{div}(F) dA = \iint_D 2 dA = 2(4\pi - \pi) = 6\pi.$$

Definition 8.10 (Divergence, Dimension 3). Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a differentiable vector field. We write $F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$. Define the **divergence** of F at the point (x, y, z) as the following real number.

$$\operatorname{div}(F)(x, y, z) = \frac{\partial}{\partial x} F_1(x, y, z) + \frac{\partial}{\partial y} F_2(x, y, z) + \frac{\partial}{\partial z} F_3(x, y, z).$$

We sometimes write $\operatorname{div}(F) = \nabla \cdot F$.

Theorem 8.11 (Divergence Theorem, Dimension 3). Let D be a region in Euclidean space \mathbb{R}^3 with boundary S . Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. Let e_n be the unit outward normal to S . Then

$$\iint_S F \cdot e_n dS = \iiint_D \operatorname{div}(F) dV.$$

Remark 8.12. In particular, the quantity on the right side only depends on the values of F on the boundary set S . Note also that the right side is a triple integral, whereas the left side is a double integral (which is also a surface integral).

Remark 8.13. Our justification for this theorem is similar to the justification for Green's Theorem. We first verify that the Divergence Theorem holds for three-dimensional boxes, which follows by an application of the single-variable fundamental theorem of calculus. A general region can then be approximated by a grid of boxes, in which the "boundary terms" mostly cancel each other.

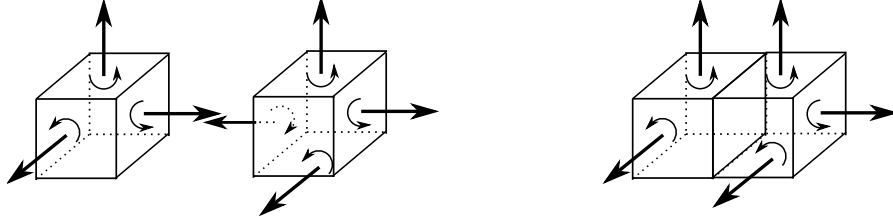


FIGURE 3. Depiction of a Proof of the Divergence Theorem

Example 8.14. Let's compute both sides of the Divergence Theorem for the following example. Let $F(x, y, z) = (x, y, z)$ be a vector field, let D denote the ball $x^2 + y^2 + z^2 \leq r^2$ where $r > 0$. Since the boundary of the ball satisfies $f(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$, the gradient of f gives the normal vector to the sphere. So, the normal vector at the point (x, y, z) is $(2x, 2y, 2z)$. So, in (x, y, z) coordinates, the unit normal vector is

$$e_n = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

So, in (x, y, z) coordinates, $F \cdot e_n = \sqrt{x^2 + y^2 + z^2} = r$. In particular, in any coordinate system on the sphere, $F \cdot e_n = r$. So, the left side of the divergence theorem is

$$\iint_S F \cdot e_n dS = \iint_S r dS = r4\pi r^2 = 4\pi r^3.$$

To compute the right side of the Divergence Theorem, note that $\text{div}(F)(x, y, z) = 3$. So,

$$\iiint_D \text{div}(F)dV = \iiint_D 3dxdydz = 3(4/3)\pi r^3 = 4\pi r^3.$$

Corollary 8.15 (Gauss's Law (One of Maxwell's Laws)). *Let D be a region in Euclidean space \mathbb{R}^3 such that D contains the origin, and let S be the boundary of D . Let $E: \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3$ be the vector field that is interpreted as the electromagnetic field of a point charge centered at the origin. That is,*

$$E(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}(x, y, z).$$

Let e_n denote the unit exterior normal to S . Then

$$\iint_S E \cdot e_n dS = 4\pi.$$

Proof. Let (x, y, z) be a point in \mathbb{R}^3 which is not the origin. Consider a large sphere U of radius r that contains S . Let D_0 denote the region in Euclidean space \mathbb{R}^3 which lies between S and U . Then D_0 does not contain the origin. We first show that $\text{div}(E)(x, y, z) = 0$. Observe

$$\begin{aligned} \text{div}(E)(x, y, z) &= (x^2 + y^2 + z^2)^{-5/2}(-3x^2 + (x^2 + y^2 + z^2) - 3y^2 + (x^2 + y^2 + z^2) - 3z^2 + (x^2 + y^2 + z^2)) \\ &= (x^2 + y^2 + z^2)^{-5/2}(-3(x^2 + y^2 + z^2) + 3(x^2 + y^2 + z^2)) = 0. \end{aligned}$$

So, by the Divergence Theorem,

$$0 = \iiint_{D_0} \operatorname{div}(E) dV = \iint_U E \cdot e_n dU - \iint_S E \cdot e_n dS.$$

It remains to show that $\iint_U E \cdot e_n dU = 4\pi$. On the sphere U , using (x, y, z) coordinates, we have $e_n = (x^2 + y^2 + z^2)^{-1/2}(x, y, z)$. So, for (x, y, z) on the sphere U (where $x^2 + y^2 + z^2 = r^2$), we have

$$E \cdot e_n = (x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-2} = r^{-2}.$$

That is, $\iint_U E \cdot e_n dU = r^{-2} \iint_U dS = r^{-2} 4\pi r^2 = 4\pi$. \square

Remark 8.16. Since the above vector field E satisfies $\operatorname{div}(E) = 0$, we similarly conclude that, if the region D does not contain the origin, and if S is the boundary of D , then $\iint_S E \cdot e_n dS = 0$. Put another way, the value of $\iint_S E \cdot e_n dS = 0$ tells us whether or not D contains the origin. So, using similar reasoning, if we have k points $(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)$ in \mathbb{R}^3 , and if for each $1 \leq i \leq k$ we define

$$E_i(x, y, z) = \|(x - x_i, y - y_i, z - z_i)\|^{-3} (x - x_i, y - y_i, z - z_i),$$

then $\sum_{i=1}^k \iint_S E_i \cdot e_n dS$ is equal to the number of points $(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)$ contained in D . In physical terms, $\iint_S (\sum_{i=1}^k E_i) \cdot e_n dS$ is the total charge enclosed by the surface S .

Remark 8.17. Gauss's law for magnetism (one of Maxwell's laws) says that if B is a vector field that is physically realizable as a magnetic field, then $\operatorname{div}(B) = 0$. From the Divergence Theorem, this is equivalent to: if D is any region in \mathbb{R}^3 with boundary S , then $\iint_S B \cdot e_n dS = 0$.

The other two of Maxwell's laws involve time derivatives, so they are a bit more complicated. All of Maxwell's laws are nevertheless closely related to Stokes' Theorem and the Divergence Theorem.

For example, let's consider Faraday's law of induction. In this case, we have vector fields $E, B: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$. That is, E represents an electric field, and B represents a magnetic field, and both fields also depend on a time variable t . For example, $E(x, y, z, t)$ has input three spatial variables (x, y, z) in \mathbb{R}^3 , and an additional time variable t in \mathbb{R} . Faraday's law asserts that $\operatorname{curl}(E) = -dB/dt$. this is equivalent to: for any surface S with boundary C , we have

$$\oint_C E \cdot T ds = -\frac{d}{dt} \iint_S B \cdot e_n dS.$$

To see the equivalence, just apply Stokes' Theorem to Faraday's law to get

$$-\frac{d}{dt} \iint_S B \cdot e_n dS = \iint_S \operatorname{curl}(E) \cdot e_n dS = \oint_C E \cdot T ds.$$

Maxwell's final law asserts that $c^2 \cdot \operatorname{curl}(B) = dE/dt$, where c is the speed of light. As before, from Stokes' Theorem, this is equivalent to: for any surface S with boundary C , we have

$$c^2 \oint_C B \cdot T ds = \frac{d}{dt} \iint_S E \cdot e_n dS.$$

To see the equivalence, just apply Stokes' Theorem to get

$$\frac{d}{dt} \iint_S E \cdot e_n dS = c^2 \iint_S \operatorname{curl}(B) \cdot e_n dS = c^2 \oint_C B \cdot T ds.$$

Exercise 8.18. Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. Verify that

$$\operatorname{div}(\operatorname{curl}(F)) = 0.$$

Exercise 8.19 (Properties of divergence). Let $g: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function, and let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. Verify:

$$\operatorname{div}(gF) = (g)\operatorname{div}(F) + (\nabla g) \cdot (F)$$

Exercise 8.20 (Green's formulas). Let $f, g: \mathbb{R}^3 \rightarrow \mathbb{R}$ be functions. Let D be a region in \mathbb{R}^3 with boundary S . Verify the following equalities

$$\begin{aligned} \bullet \iint_S f(\nabla g) \cdot e_n \, dS &= \iiint_D [f(\operatorname{div}(\nabla g)) + \nabla f \cdot \nabla g] \, dV \\ \bullet \iint_S [f(\nabla g) - g(\nabla f)] \cdot e_n \, dS &= \iiint_D [f(\operatorname{div}(\nabla g)) - g(\operatorname{div}(\nabla f))] \, dV \end{aligned}$$

Exercise 8.21. Suppose $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is twice continuously differentiable, and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Let $S(r)$ denote the sphere of radius r centered at the origin. Define

$$g(r) = \frac{1}{4\pi r^2} \iint_{S(r)} f \, dS.$$

Prove that $(d/dr)g(r) = 0$. (Hint: first change variables so that g is an integral over $S(1)$. Then, differentiate and use the divergence theorem.)

9. APPENDIX: NOTATION

\mathbb{R} denotes the set of real numbers

\in means “is an element of.” For example, $2 \in \mathbb{R}$ is read as “2 is an element of \mathbb{R} .”

$$\mathbb{R}^2 = \{(x_1, x_2): x_1 \in \mathbb{R} \text{ and } x_2 \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \{(x_1, x_2, x_3): x_1 \in \mathbb{R} \text{ and } x_2 \in \mathbb{R} \text{ and } x_3 \in \mathbb{R}\}$$

$f: A \rightarrow B$ means f is a function with domain A and range B . For example,

$$f: \mathbb{R}^2 \rightarrow \mathbb{R} \text{ means that } f \text{ is a function with domain } \mathbb{R}^2 \text{ and range } \mathbb{R}$$

Let $x = (x_1, x_2, x_3)$ and let $y = (y_1, y_2, y_3)$ be vectors in Euclidean space \mathbb{R}^3 . Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. We write $F = (F_1, F_2, F_3)$. Let $s: [a, b] \rightarrow \mathbb{R}^3$ be a parametrization of a curve C .

$x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$, the dot product of x and y

$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, the length of the vector x

$x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$, the cross product of x and y

T denotes the unit tangent vector to s , pointing in the direction s'

$$\begin{aligned} \int_C F \cdot T ds &= \int_{t=a}^{t=b} F(s(t)) \cdot \frac{s'(t)}{\|s'(t)\|} \|s'(t)\| dt \\ &= \int_{t=a}^{t=b} F(s(t)) \cdot s'(t) dt = \int_C F \cdot d\vec{s} = \int_C F_1 dx + F_2 dy + F_3 dz \\ ds &= \|s'(t)\| dt \text{ denotes the length element of } s \end{aligned}$$

If the curve C is closed, so that $s(a) = s(b)$, we write $\int_C F \cdot T ds = \oint_C F \cdot T ds$ to emphasize that C is a closed curve. Let D be a region in the plane \mathbb{R}^2 , and let $G: D \rightarrow \mathbb{R}^3$ be a parametrization of a surface S . Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a real valued function. Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field. We write $F = (F_1, F_2, F_3)$.

e_n denotes a unit normal to the surface S

$$\begin{aligned} \int_S F \cdot e_n dS &= \iint_D F(G(v, w)) \cdot \frac{(\partial G/\partial v) \times (\partial G/\partial w)}{\|(\partial G/\partial v) \times (\partial G/\partial w)\|} \|(\partial G/\partial v) \times (\partial G/\partial w)\| dv dw \\ &= \iint_D F(G(v, w)) \cdot ((\partial G/\partial v) \times (\partial G/\partial w)) dv dw = \iint_D F \cdot d\vec{S} \\ dS &= \|(\partial G/\partial v) \times (\partial G/\partial w)\| dv dw \text{ denotes the area element of } S \end{aligned}$$

$$\nabla f = \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right)$$

$$\text{curl}(F) = \nabla \times F = \left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2, \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3, \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \right)$$

$$\text{div}(F) = \nabla \cdot F = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

Let X, Y be sets, and let $f: X \rightarrow Y$ be a function. The function $f: X \rightarrow Y$ is said to be a **one-to-one correspondence** if and only if: for every $y \in Y$, there exists exactly one $x \in X$ such that $f(x) = y$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. We define

$$\det(A) = |A| = ad - bc.$$

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be a 3×3 matrix. We define

$$\det(A) = |A| = a(ei - fh) + b(fg - di) + c(dh - eg).$$

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