Please provide complete and well-written solutions to the following exercises.

Due December 5, at the beginning of class.

Assignment 11

Exercise 1. Consider the parametric curve $s\colon \mathbf{R} \to \mathbf{R}^2$ defined for any real t by

$$s(t) = (t^2 + 1, t^3 + t).$$

Find the tangent line to the curve when $t = 2$. You can express this line either in parametric form, or in the form $y = mx + b$. What is the slope of the tangent line?

Write a formula for the arc length of the curve between $t = 0$ and $t = 3$. You do not have to evaluate this integral.

Exercise 2. Using polar coordinates (r, θ), plot the function $r^2 = \sin\theta$ for $\theta \in (0, \pi)$. The end result should resemble a figure eight. Compute the area enclosed by the curve. Write a formula that computes the length of the curve [you do NOT have to evaluate this integral].

Exercise 3. Using polar coordinates (r, θ), plot the function $r = 1 + \cos\theta$. The end result should resemble an apple, or a heart. Compute the area enclosed by the curve. Compute the length of the curve.

Exercise 4. Using polar coordinates (r, θ), plot the function $r = (1/2) + \cos\theta$. Find the tangent line to the curve when $\theta = \pi/4$. You can express this line either in parametric form, or in the form $y = mx + b$. What is the slope of the tangent line?

Exercise 5. Using polar coordinates (r, θ), plot the functions $r = \sqrt{2}$ and $r^2 = 4\sin\theta$, and label their points of intersection. Find the tangent line to each curve when $\theta = \pi/4$. You can express the lines either in parametric form, or in the form $y = mx + b$. What is the slope of each tangent line?

Exercise 6. Suppose you have a heavy chain that is 50 kg and 2 meters long. The chain is resting on the ground. Calculate the work required to lift one end of the chain:

- 1 meter off the ground.
- 2 meters off the ground.
- 3 meters off the ground.

Ignore any effects of friction between the chain and the ground.

Exercise 7. Suppose there is an underground oil deposit. The oil has density ρ. The oil deposit is contained in a cavern that is the shape of a sphere of radius 100 meters. The center of the sphere is 500 meters below the surface of the earth. The sphere is completely full of oil. Determine the work require to pump all of the oil to the earth's surface.

The Exercises below are OPTIONAL. The Exercises below will NOT be covered on any quiz. These exercises are meant to help with your final exam studying.

Exercise 8 (Optional)**.** If the following limit exists, calculate it:

$$\lim_{x\to 0}(1+\sin(3x^2))^{\frac{1}{6x^2}}.$$

Exercise 9 (Optional)**.**

- Evaluate $\int_0^{\pi/12} \sin^2(3x)dx$.
- Calculate $\int \frac{\sqrt{1-x^2}}{x^2}dx$. (Hint: $\cot'(x) = -\csc^2 x$)

Exercise 10 (Optional)**.** Write the following function as a partial fraction.

$$\frac{3x^2-5x+8}{(x+1)(x^2-2x+5)}$$

Exercise 11 (Optional)**.** Decide whether or not the following integrals converge or diverge. You do not have to compute the values of the integrals.

- $$\int_0^\infty \frac{2\cos(x^4)}{x^5+\sqrt{x}+1}$$
- $$\int_0^3 \frac{10}{x(x^2+2)}$$

Exercise 12 (Optional)**.** Compute the volume of the solid of revolution obtained by rotating the region D bounded by the curves $x = 0$, $y = \ln(x)$, and $y = 0$ about the line $x = -1$.

Exercise 13 (Optional)**.** Determine whether or not the following series converge or diverge.

- $$\sum_{n=4}^{\infty}(-1)^n\frac{n}{n^2+4}$$
- $$\sum_{n=6}^{\infty}(-1)^n\left(\frac{4e^{2n}-4}{5e^n+6e^{2n}}\right)^{n-1}$$
- $$\sum_{n=2}^{\infty}2n\sin(2/n).$$
- $$\sum_{n=5}^{\infty}\ln\left(1+\frac{1}{n^3}\right).$$

Exercise 14 (Optional)**.** Define $f(x) = \frac{4}{5-x}$ for any real x.

- Show that f is an increasing function, and for any x in $[0,3]$, $f(x)$ is also in $[0,3]$.
- Let $b_0 = 5/2$, and for any $n \geq 1$ define $b_n = f(b_{n-1})$. Show that, for any $n \geq 0$, $0 \leq b_n \leq 3$. Then show that the sequence $b_0, b_1, \ldots$ is decreasing.

- Show that the sequence $b_0, b_1, \ldots$ converges to a limit L. Find the value of L.

Exercise 15 (Optional)**.** For any real x, define

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{3n+1}.$$

- Find the interval of convergence of f. That is, find all points where the sum converges.
- Find a simple formula for f (that does not involve any $\sum$ signs).

Exercise 16 (Optional)**.** Approximate the value of the integral

$$\int_0^1 \frac{x}{3+x^3} dx$$

within four decimal places of accuracy.