126 Exam 2 Solutions[1]

1. Question 1

(a) Suppose we estimate the integral of the function $f(x) = x^2$ on the interval $[0, 1]$ using the trapezoid rule with the points $0 < 1/4 < 1/2 < 3/4 < 1$. That is, we approximate the area under the curve of f by four trapezoids, whose endpoints are $0, 1/4, 1/2, 3/4, 1$ in succession. Then the trapezoid rule gives the estimate T_4, where T_4 is equal to

(i) $\frac{1}{4}\big((1/4)^2 + (1/2)^2 + (3/4)^2 + 1/2\big)$.

(b) Let $f(x) = \sin(x)$. The third order Taylor polynomial of f centered at $x = 0$ is

(ii) $x - (1/6)x^3$. Since $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = -1$, the Taylor polynomial is $xf'(0) + x^2 f''(0)/2 + x^3 f'''(0)/6 = x - x^3/6$.

(c) Suppose a square is submerged in a fluid with density ρ. The square is oriented vertically in the fluid, and its edges are two meters long. The top edge of the square touches and is parallel to the top of the fluid. Let g be the standard acceleration due to gravity. The fluid force on one side of the square is:

(iv) $\rho g \int_0^2 2h dh$. The force is $\rho g \int_0^2 f(h)h dh$. The width $f(h)$ is equal to the constant 2.

(d) If $\{a_n\}$ is a divergent sequence, then

(v) None of the above need to be true. (Consider $\{a_n\} = \{n\}$ or $\{a_n\} = \{(-1)^n\}$. The first sequence is unbounded. The second sequence is bounded, not monotonic, and $1/a_n$ does not converge.)

2. Question 2

Evaluate the following integrals. If the integral diverges, say "integral diverges."

(a) $\int_0^1 \frac{1}{x^2-1}$.

Solution. This integral diverges. Note that from the method of partial fractions, we have $1/(x^2-1) = A/(x+1)+B/(x-1)$. Solving for A, B, we get $1 = A(x-1)+B(x+1)$, so that $B = 1/2$ when $x = 1$ and $A = -1/2$ when $x = -1$. So, $1/(x^2-1) = -1/(2(x+1)) + 1/(2(x-1))$. The function $-1/(2(x+1))$ can be integrated on $[0, 1]$, but the function $-1/(2(x-1))$ cannot be integrated on $[0, 1]$. That is,

$$\int_0^1 \frac{1}{x^2-1} = -\frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{dx}{x-1} = -\frac{1}{2}[\ln(x+1)]_{x=0}^{x=1} + \frac{1}{2}\lim_{t\to 1^-}\int_0^t \frac{dx}{x-1}$$
$$= -\frac{1}{2}\ln 2 + \frac{1}{2}\lim_{t\to 1^-}[\ln|x-1|]_{x=0}^{x=t} = -\frac{1}{2}\ln 2 + \frac{1}{2}\lim_{t\to 1^-}\ln|t-1| = -\infty.$$

(b) $\int_0^{10} \frac{xdx}{\sqrt{1+x^2}}$.

Solution. We use the substitution $u = 1 + x^2$ so $du = 2xdx$, and so $\int_0^{10} \frac{xdx}{\sqrt{1+x^2}} = \int_1^{101}(1/2)u^{-1/2}du = [u^{1/2}]_{u=1}^{u=101} = \sqrt{101} - 1$.

3. Question 3

Compute the arc length of the function $f(x) = -\ln(\cos(x))$ from $x = 0$ to $x = \pi/4$.

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Solution. Note that $f'(x) = -(-\sin x)/\cos(x) = \sin(x)/\cos(x) = \tan(x)$, so $\sqrt{1+(f'(x))^2} = \sqrt{1+\tan^2(x)} = (\cos(x))^{-1}$. We compute $\int_{x=0}^{x=\pi/4} \sqrt{1+(f'(x))^2}dx = \int_{x=0}^{x=\pi/4} (\cos(x))^{-1}dx$. Now, using $(d/dx)\ln|\sec(x)+\tan(x)| = (\cos(x))^{-1}$, we get $\int_{x=0}^{x=\pi/4} \sqrt{1+(f'(x))^2}dx$
$= \ln|\sec(x)+\tan(x)|_{x=0}^{x=\pi/4} = \ln\left|\sqrt{2}+1\right| - \ln|1| = \ln\left|\sqrt{2}+1\right|$

4. Question 4

(a) Compute $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

Solution. $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.

(b) Determine whether or not the following series converges or diverges. If the series converges find its sum.

$$\sum_{n=0}^{\infty}(\sqrt{2})^n.$$

Solution. Since $\lim_{n\to\infty}(\sqrt{2})^n = \infty$, the series diverges by the divergence test.

(c) Determine whether or not the following series converges or diverges. If the series converges find its sum.

$$\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right).$$

Solution. Let $S_N = \sum_{n=1}^{N}\ln\left(\frac{n}{n+1}\right)$. This series can be written as a telescoping sum of the form

$$S_N = \ln(1)-\ln(2)+\ln(2)-\ln(3)+\ln(3)-\ln(4)+\cdots-\ln N+\ln N-\ln(N+1)$$

That is, $S_N = \ln(1)-\ln(N+1) = -\ln(N+1)$. That is $\lim_{N\to\infty}S_N = -\infty$. That is, the partial sums do not converge, so the series does not converge.

5. Question 5

(a) Using integrals, write a quantity that computes the area between the curves $x = y^3+3y+1$ and $x = 4y+1$. You do NOT have to evaluate the integrals.

Solution. We first find the points of intersection. These intersection points occur when $y^3+3y+1 = 4y+1$, i.e. when $y^3-y=0$, i.e. $y(y+1)(y-1)=0$. So, the curves intersect at $y=-1,0,1$. When $y=-1/2$, we have $y^3+3y+1=-5/8$ and $4y+1=-1$. When $y=1/2$, we have $y^3+3y+1=21/8$ and $4y+1=3$. So, when $-1\leq y\leq 0$, we have $y^3+3y+1 > 4y+1$, and when $0\leq y\leq 1$, we have $4y+1 > y^3+3y+1$. So, the area between the curves is

$$\int_{-1}^{0} y^3+3y+1-4y-1dy + \int_{0}^{1} 4y+1-y^3-3y-1dy.$$

(b) Suppose we take the region between the curves $x=y^2-3y$ and $x=2y-y^2$ and we revolve this region around the line $x=3$. Write an integral that computes the volume of the resulting solid region. You do NOT have to evaluate the integral.

Solution. We first find the intersection of the curves. Intersection occurs when $y^2-3y = 2y-y^2$, i.e. when $2y^2-5y=0$, i.e. $y(2y-5)=0$. Also, when $y=1$, we have $y^2-3y=-2$ and $2y-y^2=-1$, so $2y-y^2 > y^2-3y$ when $0<y<5/2$. So, intersection occurs when

$y = 0$ and when $y = 5/2$. Also, when $0 < y < 5/2$, the function $f(y) = y^2 - 3y = y(y-3)$ has maximum value 0, and the function $g(y) = 2y - y^2 = y(2-y)$ has maximum value 0. That is, both curves lie on the left side of the y-axis. And the curve f lies to the left of the curve g. So, using the disk method, the volume is given by

$$\pi \int_0^{5/2} (y^2 - 3y - 3)^2 - (2y - y^2 - 3)^2 dy.$$

(c) Suppose we take the curve $y = x^2 + 5$ where $1 \leq x \leq 2$ and we revolve this region around the y axis. Write an integral that computes the **surface area** of the resulting solid region. You do NOT have to evaluate the integral.

Solution. We solve for x to get $x = \sqrt{y-5}$, $1 \leq x \leq 2$, so that $6 \leq y \leq 9$. Then the x value is the radius of the rotated region, when y is fixed. The surface area formula is then

$$2\pi \int_6^9 \sqrt{y-5}\sqrt{1 + [(d/dy)\sqrt{y-5}]^2} dy.$$