Please provide complete and well-written solutions to the following exercises.

Due April 17, at the beginning of class.

Assignment 3

Exercise 1. A 70 kg skydiver jumps out of a plane. What is her terminal velocity in meters per second, assuming that $k = 10$ kg/s?

Exercise 2. Compute the following limits

$$\lim_{x\to -5} \frac{x^2 - 25}{5 - 4x - x^2}$$

$$\lim_{x\to\infty} \frac{9x - 4}{4 - 2x}.$$

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}.$$

$$\lim_{x\to\pi/2} (x - \pi/2)\tan(x).$$

Exercise 3. Find

$$\lim_{x\to\infty} \frac{e^{e^x}}{e^x}.$$

Exercise 4. What is

$$\lim_{x\to\sqrt{3}} \frac{\tan^{-1}(x) - \pi/3}{x - \sqrt{3}}?$$

Exercise 5. Using the Pythagorean Theorem, derive the following formula.

$$\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}.$$

Exercise 6 (**The shape of hanging cables and chains**)**.** Suppose we have a cable hanging between two poles of equal height. We will derive the shape of the hanging cable. That is, we will find a function $y = f(x)$, with $x = 0$ the midpoint of the cable, such that the cable follows the curve $y = f(x)$. At the outset, we assume that the function $y = f(x)$ is differentiable.

Consider a segment of the cable from $x = 0$ to $x = b > 0$. We consider this segment of cable as a single body. Then there are three distinct forces acting on this segment of cable. At $x = 0$, we assume that the curve $y = f(x)$ has a horizontal tangent. First, there is a tension force T_0 pulling the cable in the negative x direction at $x = 0$, so that this force is tangent to the curve at $x = 0$. Second, there is a tension force T pulling the cable at $x = b$, and this force is also tangent to the curve at $x = b$. Third, the force of gravity of the segment of chain from $x = 0$ to $x = b$ pulls straight down. This third force is denoted $-\rho g s(b)$, where g is the force of gravity, ρ is a constant, and $s(b)$ is the length of the chain from $x = 0$ to $x = b$.

Adding all three forces together, we must get zero, since the cable is hanging in equilibrium. Suppose at $x = b$ that the tangent line to $y = f(x)$ makes an angle θ with the x-axis. Then the sum of forces in the x-direction is $-T_0 + T\cos(\theta)$, and the sum of forces in the y-direction is $-\rho g s(b) + T\sin(\theta)$. So, $T_0 = T\cos(\theta)$ and $\rho g s(b) = T\sin(\theta)$. Since $(df/dx)(b) = \sin(\theta)/\cos(\theta)$, we have

$$\frac{df}{dx}(b) = \frac{\rho g s(b)}{T_0}. \qquad (*)$$

Now, $s(b)$ is the length of $f(x)$ from $x = 0$ to $x = b$. Let $h > 0$, and assume that s is differentiable. From the linear approximation of the derivative, we have $f(b+h) \approx f(b) + hf'(b)$. Consider the right triangle with vertices $(b, f(b))$, $(b+h, f(b+h))$, and $(b+h, f(b))$. The length of the hypotenuse of this triangle is approximately $s(b+h) - s(b)$. Also, from the Pythagorean Theorem, the length of the hypoteuse of this triangle is $\sqrt{h^2 + (f(b+h) - f(b))^2}$. Combining these facts,

$$s(b+h) - s(b) \approx \sqrt{h^2 + (f(b+h) - f(b))^2} \approx \sqrt{h^2 + h^2(f'(b))^2}.$$

So, dividing both sides by $h > 0$ we have $(s(b+h) - s(b))/h \approx \sqrt{1 + (f'(b))^2}$. Letting $h \to 0^+$, and then taking a derivative of $(*)$, we have derived the following equation

$$\frac{d^2 f}{dx^2}(b) = \frac{\rho g}{T_0}\sqrt{1 + \left(\frac{df}{dx}(b)\right)^2}. \qquad (**)$$

We can now solve for f. Show that the shape of the cable is described by

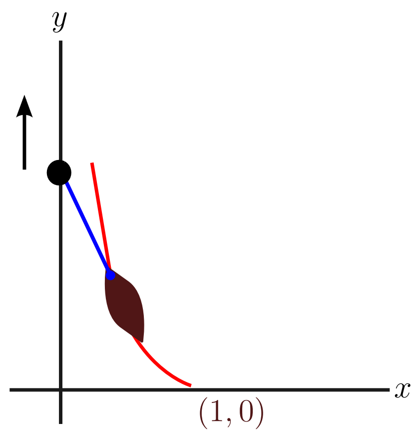
$$y = f(x) = \frac{T_0}{\rho g}\cosh\left(\frac{\rho g x}{T_0}\right).$$

The shape of the hanging cable, known as a catenary curve, also appears in architecture, such as the St. Louis Gateway Arch or Gaudi's Sagrada Familia. The idea is to freeze the hanging cable in its position, and then flip it upside down to produce an arch. Then we can repeat the derivation above to see that the forces of the arch are all the same as in the case of the chain (though the signs of the forces are flipped). Also, for a very small segment of the arch, we can essentially neglect the gravitational force exerted on this segment. So, by reviewing the above analysis, the force on any particular point in the arch will be directed along the arch itself. Therefore, the catenary arch is very stable.

Exercise 7 (**Towing an unconstrained object**)**.** Suppose I am standing on the shore of a straight river, and I am pulling on a rope of length 1 connected to the front of a canoe. I am walking at a constant speed in the positive y-direction. Suppose the canoe is in the river, and the canoe's front is at a distance x from the river shore. Suppose the initial position of the canoe is $(1, 0)$. As I move in the positive y-direction, the canoe's front is pulled along a curve denoted by $y = f(x)$. If the rope is taut, it will always be tangent to the curve $f(x)$. Consider the right triangle formed by: me on the shore, the point $(x, f(x))$ and the y-axis. Then the height of this triangle in the y-direction is $\sqrt{1 - x^2}$. Therefore,

$$\frac{df}{dx} = -\frac{\sqrt{1 - x^2}}{x}$$

Show that $f(x) = \operatorname{sech}^{-1}(x) - \sqrt{1 - x^2}$ satisfies $\frac{df}{dx} = -\frac{\sqrt{1-x^2}}{x}$.



Exercise 8. In this exercise, all velocities are measured with respect to meters per second, and $c \approx 3 \times 10^8$ meters per second is the speed of light.

Einstein's special theory of relativity implies the following fact. Suppose that I am running with velocity v_1, where we consider the earth to be fixed. Suppose I then throw a baseball at a velocity v_2, relative to my own frame of reference. Then, relative to the earth, the baseball does **not** travel with velocity $v_1 + v_2$. Relative to the earth, the baseball travels with velocity V, where

$$\tanh^{-1}(V/c) = \tanh^{-1}(v_1/c) + \tanh^{-1}(v_2/c).$$

Using a calculator or other electronic aid, find the velocity V of the baseball relative to the earth if $v_1 = 2 \times 10^8$ and $v_2 = 2.5 \times 10^8$.

Exercise 9. Using integration by parts, compute the following integrals

- $\int t^2 e^{4t} dt$.
- $\int x \sin(x/2) dx$.
- $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx$.
- $\int_2^4 x (\ln x)^2 dx$.

Exercise 10. Compute the following integrals

- $\int \sin^4(x) \cos^2(x) dx$.
- $\int \cos(x) \sin^{111}(x) dx$.
- $\int \tan^3(x) \sec^2(x) dx$.

Exercise 11. Using trigonometric substitution, compute

$$\int \frac{x^2}{\sqrt{9 - x^2}}\, dx.$$

Exercise 12. Using trigonometric substitution, compute

$$\int \frac{dx}{\sqrt{25x^2 - 4}}.$$

Exercise 13. Compute the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.