Please provide complete and well-written solutions to the following exercises.

(No due date, though the quiz on April 7th or April 9th will be based on this homework.)

Assignment 2

Exercise 1. Simplify the following expressions, to the best of your ability.

- $\log_{64} 2$.
- $\log_8 4 + \log_4 2$
- $\ln 1$
- $8^{4\log_8(3)}$

Exercise 2. Solve for x:

$$\ln(x^2+1) - 3\ln x = \ln 2.$$

Exercise 3. Differentiate the function

$$y = \frac{1+\ln t}{t}.$$

Exercise 4. Evaluate the following integral

$$\int_3^5 \frac{2\ln x}{x} dx.$$

Exercise 5. Evaluate

$$\int_{-9}^{-3} \frac{1}{t} dt.$$

Exercise 6. Using logarithmic differentiation (i.e. using $(d/dx)\ln f(x) = f'(x)/f(x)$, so that $f'(x) = f(x)(d/dx)\ln f(x)$, when f is positive), differentiate the function

$$f(x) = \sqrt{\frac{x(x+1)}{(2x+2)(3x+1)}}, \qquad x > 0.$$

Exercise 7. Differentiate the function

$$f(x) = e^{x^x}.$$

Exercise 8. A quantity $P(t)$ satisfies $P(t) = Ce^{kt}$, where t is measured in years. Find a formula for $P(t)$ assuming that the doubling time of P is 7 years, and $P(0) = 100$.

Exercise 9. In class, we discussed "first order" reactions in chemistry, and we noted that if $y(t)$ is the concentration of a chemical at time t such that $y'(t) = ky(t)$, then $y(t) = y_0 e^{kt}$. In chemistry, a "second order" reaction satisfies $dy/dt = k(y(t))^2$. So, the concentration of the chemical squared is proportional to the rate of change of the concentration. If $y(t)$ is a second order reaction and if $y_0 = y(0) \neq 0$, verify that we must have

$$y(t) = \frac{1}{y_0^{-1} - kt}.$$

Exercise 10. Scientists can determine the age of ancient objects by the method of **radiocarbon dating**. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of Carbon-14, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates Carbon-14 through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of Carbon-14 begins to decrease through radioactive decay. Therefore the level of radioactivity of the carbon must also decay exponentially.

A parchment fragment was discovered that had about 74% as much Carbon-14 radioactivity as does plant material on the earth today. Estimate the age of the parchment.

Exercise 11 (**Newton's Law of Cooling**). Suppose $y(t)$ is the temperature of an object at time t. If an object is of a different temperature than its surroundings, then the rate of change of the object's temperature is proportional to the difference of the temperature of the object and the temperature of the surroundings. That is, if Y denotes the temperature of the surroundings, and if $y(0) = y_0 \neq Y$, then there exists a constant $k > 0$ such that

$$y'(t) = -k(y(t) - Y).$$

Note that if $y(0) < Y$, then $y'(0) > 0$, so that the temperature of y is increasing to the environment's temperature. And if $y(0) > Y$, then $y'(0) < 0$, so that y is decreasing to the environment's temperature.

Let $f(t) = y(t) - Y$. Verify that $f'(t) = -kf(t)$. Conclude that $f(t) = y(t) - Y = (y_0 - Y)e^{-kt}$. That is, we have Newton's Law of cooling:

$$y(t) = Y + (y_0 - Y)e^{-kt}.$$

Exercise 12. Suppose a hard-boiled egg is at 98^oC and it is put into a larger tank of water at the constant temperature of 18^oC. After 5 minutes, the egg's temperature is 38^oC. What is the total length of time that it takes for the egg to go from 98^oC to 20^oC?

Exercise 13. The exponential growth model for bacteria is a bit unrealistic, since after a while, the bacteria are limited by their environment and food supply. We therefore consider the **logistic growth** model. Suppose $y(t)$ is the amount of bacteria in a petri dish at time t and $k > 0$ is a constant. Let C be the maximum possible population of the bacteria. We model the growth of the bacteria by the formula

$$y'(t) = ky(t)(C - y(t)), \qquad y(0) = y_0$$

So, when y is small, $y'(t)$ is proportional to y. However, when y becomes close to C, y' becomes very small. That is, the rate of growth of bacteria is constrained by the environment.

- Verify that the following function satisfies the above differential equation.

$$y(t) = \frac{C}{1 + (Cy_0^{-1} - 1)e^{-kt}}.$$

- Plot the function $y(t)$. (What are the limits of y as t goes to $+\infty$ and $-\infty$?)
- Find out where $y'(t)$ is the largest. (Hint: find the maximum of the function of y: $ky(C - y)$.)

The latter observation explains the "J-curve" scare for human population growth in the 1980s. At this point in time, many people were afraid that the human population would grow too large for the earth to support us. However, it seems that we were simply observing the maximum possible growth rate of the human population at this time (if we believe that the logistic growth models the human population reasonably well).