Please provide complete and well-written solutions to the following exercises.

Due June 5, at the beginning of class.

Assignment 10

Exercise 1. Find the Maclaurin series of

$$(x^2+1)^2.$$

Exercise 2. Find the Maclaurin series of

$$e^x \ln(1+x).$$

Exercise 3. Show that the following series converges to zero.

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$$

Exercise 4. Express the following integral as an infinite series for $|x| < 1$

$$\int_0^x \ln(1+t^2)dt.$$

Exercise 5. The following integral often arises in probability theory, in relation to diffusions, Brownian motion, and so on.

$$F(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Using a Taylor series for e^{-t^2}, find a Taylor series for F. Then, find the radius of convergence of this series. Finally, compute $F(1/\sqrt{2})$ to four decimal places of accuracy, and then compute $F(2/\sqrt{2})$ to two decimal places of accuracy. The answers should remind you of the concept of standard deviation. (F is also known as a bell curve, or the error function. Optional: give an estimate for $F(3/\sqrt{2})$)

Exercise 6. Let $i = \sqrt{-1}$. Using the Maclaurin series for $\sin(x)$, $\cos(x)$ and e^x, verify Euler's identity

$$e^{ix} = \cos(x) + i\sin(x).$$

In particular, using $x = \pi$, we have

$$e^{i\pi} + 1 = 0.$$

Also, use Euler's identity to prove the following equalities

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

In particular, we finally see that the hyperbolic sine and cosine functions are exactly the usual sine and cosine functions, evaluated on imaginary numbers.

$$\cos(x) = \cosh(ix).$$
$$\sin(x) = \sinh(ix)/i.$$

Exercise 7. Euler's identity can be used to remember all of the multiple angle formulas that are easy to forget. For example, note that

$$\cos(2x) + i\sin(2x) = e^{2ix} = (e^{ix})^2 = (\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + 2i\sin(x)\cos(x).$$

By equating the real and imaginary parts of this identity, we therefore get

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\sin(2x) = 2\sin(x)\cos(x).$$

Derive the triple angle identities in this same way, using $e^{3ix} = (e^{ix})^3$.

Exercise 8. This exercise justifies the identity

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

by following Euler's original proof.

- It turns out that we can write $\sin(x)$ as an infinite product of its zeros. (We can only do this very rarely.)
$$\begin{aligned}\frac{\sin(\pi x)}{\pi x} &= \cdots \left(1 - \frac{x}{3}\right)\left(1 - \frac{x}{2}\right)\left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{1}\right)\left(1 + \frac{x}{2}\right)\left(1 + \frac{x}{3}\right)\cdots \\ &= \left(1 - x^2\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\left(1 - \frac{x^2}{4^2}\right)\cdots.\end{aligned}$$
The term on the right side can be expressed as a power series in x. The constant term is 1, and there is no x term. What is the x^2 term? (You are allowed to rearrange terms as you wish.)
- What is the x^2 term in the usual Maclaurin series for $\dfrac{\sin(\pi x)}{\pi x}$?
- Equate both of these x^2 terms of $(\sin(\pi x))/\pi x$

Exercise 9. Find the sum of the infinite series

$$1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \frac{2}{10^7} + \frac{3}{10^8} + \frac{7}{10^9} + \cdots.$$

Exercise 10. Using Maclaurin series, evaluate the following limit

$$\lim_{x\to 0} \frac{\cos(x) - 1 + x^2/2}{x^4}.$$