

Please provide complete and well-written solutions to the following exercises.

Due November 20, at the beginning of class.

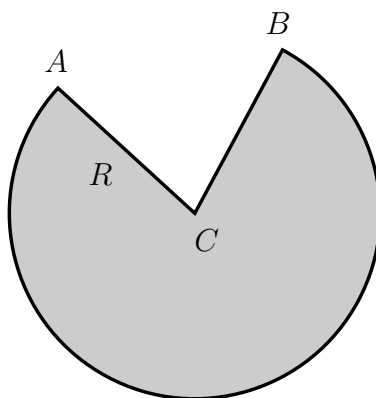
Assignment 6

Exercise 1. Find two numbers whose difference is 100 and whose product is a minimum.

Exercise 2. Suppose 1200 cm^2 of material is available to make a box with a square base and an open top. Find the largest possible volume of the box.

Exercise 3. Find the point on the line $y = 2x + 3$ that is closest to the origin.

Exercise 4. Suppose we have a circular piece of paper with center C and radius R . Let A, B be two points on the boundary of the circle. A cone-shaped paper drinking cup is made from the circular piece of paper by cutting out a sector CAB and joining the edges CA and CB . Find the maximum capacity of such a cup.



Exercise 5. For a fish swimming at a speed v relative to the water, the energy expenditure per unit of time is proportional to v^3 . It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current of speed u ($u < v$), then the time required to swim a distance L is $L/(v - u)$, and the total energy E required to swim the distance L is given by

$$E(v) = av^3 \frac{L}{v - u}$$

Here a is an arbitrary constant.

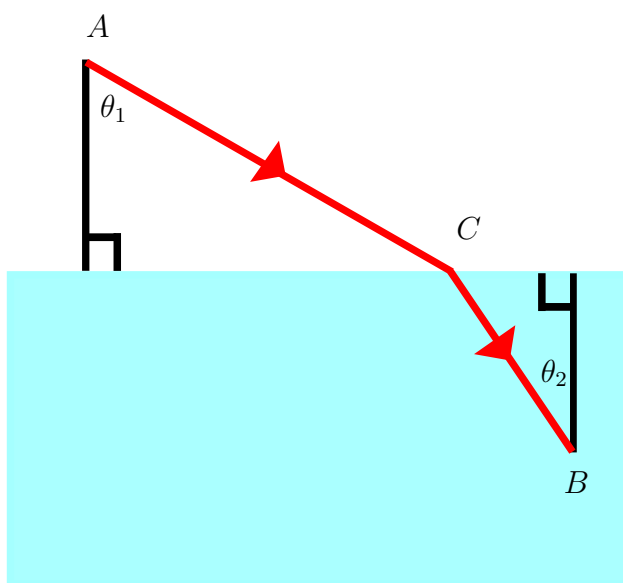
- Determine the value of v that minimizes E .
- Sketch the graph of E .

Note: This result has been verified experimentally. Migrating fish swim against a current at a speed 50% greater than the speed of the current.

Exercise 6. Let v_1 be the velocity of light in air and let v_2 be the velocity of light in water. Suppose a light ray travels from point A in the air to a point B in the water. Let C be the point where the light ray first hits the water. At this point, the light bends, and then continues in a straight line. According to Fermat's Principle, the ray of light has traveled along the path ACB that minimizes the elapsed travel time between the points A and B . Let θ_1 be the angle of incidence and let θ_2 be the angle of refraction. From Fermat's Principle, derive Snell's Law:

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

As the light travels from A , θ_1 is the angle that the ray makes with a vertical line. Also, as the light hits B , θ_2 is the angle that the ray makes with a vertical line.



Exercise 7. (Branching angles for blood vessels and pipes) When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction can be minimized along the section AOB shown in Figure 1. In this problem, the pipe AC is fixed, and the point B is fixed. The point O is variable. So, we know that the larger pipe AC branches at O into the smaller pipe OB . We just allow the point O to vary. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent fluid flow is proportional to the length of the path. This energy loss is also inversely proportional to the fourth power of the radius of the pipe. Let k be a constant, let d_1 be the length of AO , let d_2 be the length of OB , let θ be the angle BOC , let R be the radius of the pipe AC , and let r be the radius of the pipe OB . Then the energy loss along AO is $(kd_1)/R^4$, and the energy loss along OB is $(kd_2)/r^4$. We are required to minimize the energy loss along the path AOB . So, we need to minimize

$$L = k \frac{d_1}{R^4} + k \frac{d_2}{r^4}$$

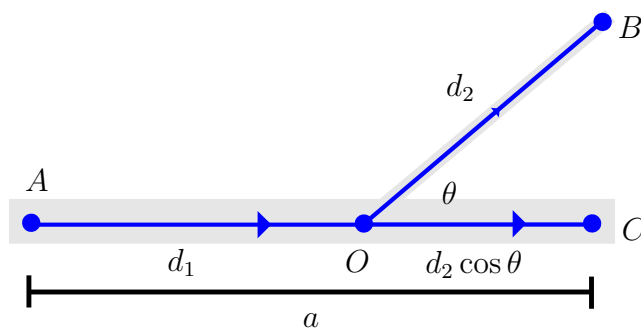


FIGURE 1.

To review, $AC = a$ and $BC = b$ are fixed. By the definitions of our constants, we have

$$d_1 + d_2 \cos \theta = a, \quad \text{and} \quad d_2 \sin \theta = b.$$

So, $d_2 = b/\sin \theta$ and $d_1 = a - d_2 \cos \theta = a - b \cos \theta/\sin \theta$. We can therefore express L as a function of θ as follows

$$L = L(\theta) = k \left(\frac{a - b \frac{\cos \theta}{\sin \theta}}{R^4} + \frac{b}{r^4 \sin \theta} \right).$$

- (a) Show that the critical point of θ for which $dL/d\theta = 0$ is given by

$$\cos \theta_c = \frac{r^4}{R^4}.$$

- (b) If the ratio of the pipe radii is $r/R = 5/6$, estimate (in degrees) the optimal branching angle given in part (a).

The analysis used here can also be used to explain the angles at which arteries branch in an animal's body.

Exercise 8. A cabinetmaker uses mahogany to produce 5 furnishings each day. Each delivery of one container of wood costs \$5000, and storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 furnishing. How much material should be ordered each time and how often should the material be delivered to minimize her average daily cost in the production cycle between deliveries? (You can consider one container of wood to have an unlimited capacity, and the storage cost of one day is equal to the number of units of wood in the shop at the beginning of the day.)

Exercise 9. (How we cough) When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity of air, and whether the trachea really contracts that much when we cough.

Let r_0 be the rest radius of the trachea in centimeters, and let c be a positive constant whose value depends in part on the length of the trachea. Under reasonable assumptions about how the air near the wall is slowed by friction, the average air flow velocity v can be modeled

by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \quad \frac{r_0}{2} \leq r \leq r_0.$$

Show that v is greatest when $r = (2/3)r_0$. That is, the velocity is greatest when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

Exercise 10. The average car achieves its best fuel efficiency at a speed of around 50 or 55 miles per hour. Driving faster than this speed can drastically reduce fuel efficiency, as we now show.

The fuel efficiency $f(v)$ of a car (in miles per gallon) traveling at a velocity v (in miles per hour) can be roughly modelled as

$$f(v) = 200,000 \frac{v}{v^3 + 2 \cdot 50^3}$$

When v is small, resistance from the tires, the mechanical aspects of the engine, etc. contribute to the inefficiency. When v is large, resistance from air also becomes a factor.

Find the maximum fuel efficiency of the car. Compare this efficiency to the values $v = 60$, $v = 70$ and $v = 80$. (Due to air friction and other frictional forces, driving faster is often less efficient.)