Digest 7

(A compilation of emailed homework questions, answered around Wednesday.)

Question. (From a student): Other than the practice exams, etc., is there any material we should study for the exam?

Answer. As before, there is no homework this week, so you can study for the exam. That said, graph sketching and optimization were not really covered in the last homework. So, it might be helpful to do at least a few problems from homework 6 in preparation for the exam, since the exam will certainly concern both graph sketching and optimization.

Question. (From a student): Question 5 on the last practice exam from the previous digest was hard, can you provide a solution?

Answer. Sure. We have a function $f\colon \mathbf{R} \to \mathbf{R}$ such that f, f' and f'' exist for all real x. Also, $f(0) > 0$, $f'(0) = 0$ and $f''(x) > 1$ for all x.

(a) Find an equation for the critical points of $f(x)/x$.

Let $g(x) = f(x)/x$. Then $g'(x) = \frac{xf'(x)-f(x)}{x^2}$. So, if $x \neq 0$ and if $g'(x) = 0$, then $xf'(x) = f(x)$, so that $f'(x) = f(x)/x = g(x)$. Since $f(0) > 0$, $g(x)$ is undefined at $x = 0$.

(b) Describe the graph of f. If a satisfies $g'(a) = 0$, then find the y-intercept of the tangent line of f at $x = a$.

Since $f''(x) > 1$ for all x, we know that $f'(x)$ is an increasing function. That is, $f'(x) = 0$ for exactly only point a. So, the function f is concave up, f is decreasing when $x < a$, and f is increasing after $x > a$. Also, since $f'(0) = 0$ is given information, we know that 0 is the only critical point of f. So, $x = 0$ is the global minimum of f. We conclude that $f(x) \geq f(0) > 0$ for all x.

If $g'(a) = 0$, then $f'(a) = f(a)/a$, so the tangent line for f at a is $y(x) = f(a)+f'(a)(x-a) = f(a) + (f(a)/a)(x - a) = f(a)(1 + (x - a)/a)$. So, when $x = 0$, we have $y(0) = 0$. That is, the tangent line of f at a intersects the y-axis at $y = 0$.

(c) How many critical points does $f(x)/x$ have?

The function $f(x)/x$ has exactly two critical points when $x \neq 0$. These critical points arise from our discussion in part (b). If $g'(a) = 0$, then the tangent line of f intersects the origin $(x, y) = (0, 0)$. Since f is concave up (that is, since f resembles a function such as $1 + x^2$), as the point a varies along the real line, the tangent line to f at a will intersect the origin exactly two times; once for some $a < 0$, and again for some $a > 0$. (For example, if we had $f(x) = 1 + x^2$, then the tangent line intersects the origin when $a = 1$ and $a = -1$.)

(d) [bonus points] What kind of extrema are the critical points of $f(x)/x$? That is, are they local maxima or local minima? (I found this problem to be pretty tricky myself. Maybe there is a simpler solution, but I came up with the following using the Mean Value Theorem.)

We claim that $f'(x) \geq x$ for all $x > 0$. If not, then $f'(a) < a$ for some $a > 0$. So, the Mean Value Theorem implies that $\frac{f'(a)-f'(0)}{a-0} = f''(c)$ for some $c \in (0, a)$. Since $f'(0) = 0$, this says $f'(a)/a = f''(c)$. Since $f''(x) > 1$ for all x, we have $f'(a)/a > 1$, so that $f'(a) > a$, a contradiction. We conclude that $f'(x) \geq x$ for all $x > 0$. We now show that $f(x) \geq x^2/4 + f(0)$ for all $x > 0$. If not, then $f(a) < a^2/4 + f(0)$ for some $a > 0$. The Mean Value Theorem implies that $\frac{f(a)-f(a/2)}{a-a/2} = f'(c)$ for some $c \in (a/2, a)$. Since $f'(x) \geq x$ for all $x > 0$, we have $(f(a) - f(a/2))/(a/2) \geq c \geq a/2$, so that $f(a) \geq a^2/4 + f(a/2) \geq a^2/4 + f(0)$, a contradiction. We conclude that $f(x) \geq x^2/4 + f(0)$ for all $x > 0$.

Using similar reasoning, we can conclude that $f(x) \geq x^2/4 + f(0)$ for all $x < 0$. (We first show that $f'(x) \leq x$ for all $x < 0$, and then show that $f(x) \geq x^2/4 + f(0)$.)

Since $f(x) \geq f(0) > 0$ for all x as we showed in part (b), we know that $\lim_{x\to 0^-} f(x)/x = -\infty$ and $\lim_{x\to 0^+} f(x)/x = +\infty$. Also, using $f(x) \geq x^2/4 + f(0)$, we know that $\lim_{x\to\infty} f(x)/x = +\infty$ and $\lim_{x\to-\infty} f(x)/x = -\infty$. So, the critical point of $f(x)/x$ that occurs when $x < 0$ is a local maximum (since $f(x)/x$ goes to $-\infty$ at the endpoints of the interval $(-\infty, 0)$). And the critical point of $f(x)/x$ that occurs when $x > 0$ is a local minimum (since $f(x)/x$ goes to $+\infty$ at the endpoints of the interval $(0, \infty)$).

Question. (From a student): Will there be any extra credit assignments?

Answer. No. Since the class is curved, if I made an extra credit assignment and everyone did it, then no one's grade would change. If you would like, you could think of a hard question on a homework or an exam as extra credit, since if you put in extra effort to get it right, then it will probably increase your grade.