31A: CALCULUS, FIRST QUARTER, FALL 2015

STEVEN HEILMAN

Contents

Date: December 3, 2015.

1. Introduction

The ability of Calculus to describe the world is one of the great triumphs of mathematics. Below, we will briefly describe some of these successful applications. Some of these applications will be discussed in the second half of this course, and others may be found in your future endeavors.

For example, calculus is very closely related to **probability**, which itself has applications to **statistics** and **algorithms**. For example, the first generation of Google's search technology came from ideas from probability theory. In **economics**, optimization is often used to e.g. maximize profit margins, or to find optimal strategies in **game theory**. Also, the ideas of single variable calculus are developed and generalized within financial mathematics to e.g. **stochastic calculus**. Signal processing and **Fourier analysis** provide some nice applications within many areas of science. For example, our cell phones use Fourier analysis to compress voice signals.

In **physics**, many models of the real world use solutions of **differential equations**. These equations involve the slopes and shape of functions, and their solutions describe the behavior of many physical systems. For example, the famous Navier-Stokes equations of fluid dynamics are expressed in this language. Solutions of the Navier-Stokes equations show us how water behaves, though these equations really just state Newton's second law. Also, **Einstein's Theory of General Relativity** uses a version of Calculus, though geometry is needed here as well.

In **mathematics** itself, the fundamental concepts of Calculus reappear in many places, some of which have already been described above. Also, Calculus serves as the foundation of **probability**, which itself serves as the foundation of **statistics**. For example, to prove that a large number of numerical data samples have the distribution of a bell curve, one can use tools from Calculus. As another example, we can repeat Calculus for a single real variable by using a single complex variable, and we get the beautiful subject of **Complex analysis**.

For a complete understanding of **biology**, you need to understand Calculus. For example, suppose someone is given an intravenous drug which is administered at a certain rate. If we know the volume of fluid in the body, the concentration of the administered drug, and the rate of flow of the drug into the body, then we can use Calculus to model the concentration of drug in the person's body. These calculations are done using differential equations. If one wants to fully understand how MRIs and CT-scans work, one needs **multivariable calculus**. Many of the concepts in multivariable calculus originate in single variable calculus, though there are some new things that are needed.

Using differential equations, one can derive some of the equations for chemical concentrations that were used in our high school **chemistry** classes. Also, the ideas of Calculus are used in more advanced chemistry subjects, including **quantum mechanics**.

Even though **computer science** often deals with discrete problems, many of the ideas of Calculus arise in computer science, and sometimes these ideas arise in unexpected ways. For example, the methods used to encode music onto CDs and MP3s use Calculus, with some additional tools from **Fourier analysis**. These same tools compress image and video data for JPEGs and MPEGs, respectively.

The acts of thinking rigorously and using **logic** in our reasoning should become common in this course. We want to transmit one of the great intellectual achievements of humanity, just as we pass down great literature, art and philosophy to future generations.

1.0.1. *A Brief History of Calculus.* The rudiments of Calculus can be traced through many ancient cultures including those of Greece, China and India. Calculus in its modern form is generally attributed to Newton and Leibniz in the late 1600s. Newton was mainly motivated by applying Calculus to physics. However, Leibniz invented most of the notation we use today. Even though this topic is now taught in high school and college, it is still over 300 years old. In this course, perhaps we will try to give you glimpses of what happened in the ensuing centuries.

1.0.2. *The Calculus Paradigm.* Mathematics before Calculus usually involves algebra, trigonometry and some planar geometry. The concepts in Calculus are very different from concepts that are learned before, so the following "paradoxes" are meant to help in understanding the new concepts. These paradoxes are known as Zeno's Paradoxes.

Paradox 1.1. In a footrace, suppose a slower runner is in front of a quicker runner. When the quicker runner reaches any point in the race, the slower runner was already at that point in the past. Therefore, the quicker runner can never overtake the slower runner.

Paradox 1.2. Suppose I want to walk through a doorway, and I am standing a meter away from the door. At some point I am at a half meter away from the door, then at another point I am at a quarter of a meter away, then at another point I am at an eighth a meter away, and so on. Therefore, I can never make it through the doorway.

Paradox 1.3. Suppose I shoot at arrow at a target. At any given moment, the arrow occupies a fixed position in space. However, in order for an object to move, it cannot sit in one place. So, the arrow must have no motion at all. The arrow is motionless.

The first two paradoxes are somewhat similar. In the first paradox, we know from empirical observation that a quicker runner can overtake a slower runner. And in order to resolve the paradox, we note that the total time that the quicker runner remains behind the slower runner is finite. Similarly, in the second paradox, I know that it only takes a finite amount of time to pass through a door. Paradox 1.2 seems to occur since I am subdividing the one meter that I travel into an infinite number of smaller steps. Zeno seems to object, saying that an infinite number of subdivisions cannot occur.

In mathematical terms, Zeno objects to the assertion $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$, because there does not appear to be a rigorous way to think about an infinite sum of numbers. By using a limit, we can actually speak rigorously about an infinite sum of numbers, thereby resolving the paradox.

The third paradox is a bit different from the other two. Zeno is really questioning the meaning of an instant of time. How can we rigorously discuss the instantaneous speed of an object? Below, we will use limits to define derivatives, and these derivatives give a rigorous meaning to instantaneous velocity, thereby resolving the paradox.

1.1. **The Notion of a Limit.**

Definition 1.4 (Intuitive Definition of a Limit). Let $f\colon \mathbb{R} \to \mathbb{R}$, and let $x, a, L \in \mathbb{R}$. We say that f has limit L as x approaches a and write

$$\lim_{x\to a} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a, with $x \neq a$.

Definition 1.5 (**Formal Definition of a Limit**)**.** Fix $a \in \mathbb{R}$ and let x be a variable. We write $\lim_{x\to a} f(x) = L$ if the following occurs.

$$\begin{aligned} &\text{For all } \varepsilon > 0, \text{ there exists } \delta = \delta(\varepsilon) > 0, \text{ such that:} \\ &\text{if } 0 < |x-a| < \delta, \text{ then } |f(x) - L| < \varepsilon. \end{aligned} \tag{1}$$

Put another way, no matter how small I want $|f(x) - L|$ to be, I can always choose a small region around a that is so small such that $f(x)$ is really close to L in this region.

Example 1.6. Let $f(x) = x$ and let $a = 1$. Then $\lim_{x\to a} f(x) = a = 1$.

Example 1.7. Let $f(x) = x^2$ and let $a = 2$. Then $\lim_{x\to a} f(x) = a^2 = 4$.

Remark 1.8. The limit $\lim_{x\to a} f(x)$ does not depend on the value of f at a.

Example 1.9. Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Then $\lim_{x\to 0} f(x) = 1$.

Definition 1.10 (**Intuitive Definition of One-Sided Limits**)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$, and let $x, a, L \in \mathbb{R}$. We say that f has limit L as x approaches a from the left and write

$$\lim_{x\to a^-} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a, with $x < a$.

We say that f has limit L as x approaches a from the right and write

$$\lim_{x\to a^+} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a, with $x > a$.

Remark 1.11. $\lim_{x\to a} f(x)$ exists if and only if both one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and are equal.

The limit may at first look a bit silly[1], since for a polynomial, we can just plug in the function value and the result agrees with the limit. So why are we defining a limit anyway? First of all, we need limits in order to define tangent lines to functions (derivatives), and derivatives are one of the extremely important concepts in Calculus. Second of all, there are some subtleties to the definition that may not yet be apparent.

For example, the limit of a function may not always exist. This issue is important because we will see that the derivatives of some functions may not exist. So, there may be no reasonable way to talk about the instantaneous speed of certain trajectories.

Example 1.12 (**A jump discontinuity**))**.** Define $H\colon \mathbb{R} \to \mathbb{R}$ (the Heaviside function) by the formula

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

We claim that $\lim_{x\to 0^-} H(x) = 0$ and $\lim_{x\to 0^+} H(x) = 1$. Therefore, $\lim_{x\to 0} H(x)$ does not exist.

[1]Historically, the definition (1) was invented over a century after Newton and Leibniz invented Calculus. The approach of Newton and Leibniz was not quite rigorous by modern standards.

To see this, note that if $x < 0$, then $H(x) = 0$. That is, $H(x)$ is always close to 0 whenever $x < 0$. We conclude that $\lim_{x \to 0^-} H(x) = 0$.

Similarly, if $x > 0$, then $H(x) = 1$. That is, $H(x)$ is always close to 1 whenever $x > 0$. Therefore, $\lim_{x \to 0^+} H(x) = 1$.

Both one-sided limits must be equal in order for $\lim_{x \to 0} H(x)$ to exist. Since we know that $\lim_{x \to 0^+} H(x) = 1 \neq 0 = \lim_{x \to 0^-} H(x)$ we conclude that $\lim_{x \to 0} H(x)$ does not exist.

Remark 1.13. If a function f becomes arbitrarily large as $x \to a$, the limit $\lim_{x \to a} f(x)$ does not exist, but we still write $\lim_{x \to a} f(x) = \infty$. If a function f has $|f(x)|$ arbitrarily large and $f(x) < 0$ as $x \to a$, the limit $\lim_{x \to a} f(x)$ does not exist, but we still write $\lim_{x \to a} f(x) = -\infty$.

Example 1.14 (**A singularity**)**.** Define $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We claim that $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$. Therefore, $\lim_{x \to 0} f(x)$ does not exist.

Note that if $x > 0$, then $f(x) > 0$. And, for example, if $x = 1/n$ where n is a positive integer, then $f(x) = n$ becomes arbitrarily large as n becomes large. Therefore, $\lim_{x \to 0^+} f(x) = \infty$. Since $\lim_{x \to 0^+} f(x)$ does not exist, we already know that $\lim_{x \to 0} f(x)$ does not exist.

Similarly, if $x < 0$, then $f(x) < 0$. And if $x = -1/n$ where n is a positive integer, then $f(x) = -n$ becomes negative and large as n becomes large. Therefore, $\lim_{x \to 0^-} f(x) = -\infty$.

Example 1.15 (**Infinite oscillation**)**.** Define $f : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We claim that $\lim_{x \to 0} f(x)$ does not exist. To see this, let n be a positive integer and consider the points $1/(2\pi n)$ and $1/(2\pi n + \pi)$. If n is large, both of these points can become arbitrarily close to zero. However, $f(1/(2\pi n)) = \cos(2\pi n) = 1$, while $f(1/(2\pi n + \pi)) = \cos(2\pi n + \pi) = -1$. So, f never becomes close to any value L as $x \to 0$. So, $\lim_{x \to 0} f(x)$ does not exist.

1.2. **Calculating Limits.** In order to manipulate limits, we need to be able to apply some simple operations to them. The following statements summarize some ways that we can manipulate limits.

Proposition 1.16 (**Limit Laws**)**.** *Let* $f, g : \mathbb{R} \to \mathbb{R}$. *Let* $a, c \in \mathbb{R}$. *Assume that* $\lim_{x \to a} f(x)$ *exists and* $\lim_{x \to a} g(x)$ *exists. Then*

- (i) $\lim_{x \to a}(f(x) + g(x)) = (\lim_{x \to a} f(x)) + (\lim_{x \to a} g(x))$.
- (i') $\lim_{x \to a}(f(x) - g(x)) = (\lim_{x \to a} f(x)) - (\lim_{x \to a} g(x))$.
- (ii) $\lim_{x \to a}[cf(x)] = c(\lim_{x \to a} f(x))$.
- (iii) $\lim_{x \to a}[(f(x))(g(x))] = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$.
- (iv) *If* $\lim_{x \to a} g(x) \neq 0$, *then* $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$.

Example 1.17. We know that $\lim_{x \to 2} x = 2$, so we know that $\lim_{x \to 2} x^2 = (\lim_{x \to 2} x)^2 = 2^2 = 4$, using Limit Law (*iii*).

Example 1.18. Since we know that $\lim_{x\to 2} x^2 = 4$ and $\lim_{x\to 2} x = 2$, we then know that $\lim_{x\to 2} x^3 = (\lim_{x\to 2} x^2)(\lim_{x\to 2} x) = 2^2 2 = 8$, using Limit Law (iii).

Example 1.19. Since $\lim_{x\to 0}(x+1) = 1$, and $\lim_{x\to 0}(x^2+3) = 3$, we have

$$\lim_{x\to 0} \frac{x+1}{x^2+3} = \frac{1}{3}.$$

Example 1.20. We know that $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0} x^{-1}$ does not exist. So, the following equality doesn't make any sense:

$$1 = \lim_{x\to 0} 1 = \lim_{x\to 0}(x/x) \stackrel{?}{=} (\lim_{x\to 0} x)(\lim_{x\to 0} x^{-1}).$$

However, the assumptions of the Limit Laws are not satisfied, so we have not found a contradiction within the Limit Laws.

1.3. **Continuity.**

Definition 1.21 (Intuitive Definition of Continuity). Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. We say that f is **continuous** if we can draw f with a pencil, without lifting the pencil off of the paper.

Alternatively, f is continuous at a if f does not "jump around too much" around a.

Definition 1.22 (Formal Definition of Continuity). Let $f\colon \mathbb{R} \to \mathbb{R}$ let $a \in \mathbb{R}$. We say that f is **continuous at** a if the following three conditions are satisfied

(i) $\lim_{x\to a} f(x)$ exists
(ii) a is in the domain of f
(iii) $\lim_{x\to a} f(x) = f(a)$

We say that a function f is **continuous** if f is continuous on all points of its domain. If f is not continuous at a, we say that f is **discontinuous** at a.

Example 1.23. Let $f(x) = x$. Then f is continuous, since $\lim_{x\to a} f(x) = a = f(a)$ for all points $a \in \mathbb{R}$.

Example 1.24 (Removable Discontinuity). Let

$$f(x) = \begin{cases} 1 & \text{if}\, x \neq 0 \\ 0 & \text{if}\, x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 1$. That is $\lim_{x\to 0} f(x) \neq f(0)$. So f is discontinuous at 0. However, if we redefined f to be equal to 1 at 0, then f would be continuous at 0. For this reason, we say that f has a **removable discontinuity** at 0.

Example 1.25 (Jump Discontinuity)). Recall we defined $H\colon \mathbb{R} \to \mathbb{R}$ where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

We showed that $\lim_{x\to 0} H(x)$ does not exist. So, $H(x)$ is discontinuous at $x = 0$.

Example 1.26 (**A singularity**)**.** Define $f\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We showed that $\lim_{x\to 0} f(x)$ does not exist. So, $f(x)$ is discontinuous at $x = 0$. However, f is actually continuous at any nonzero point, by the limit law for quotients. If $a \neq 0$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a}(1/x) = 1/(\lim_{x\to a} x) = 1/a = f(a).$$

Example 1.27 (**Infinite oscillation**)**.** Define $f\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = \begin{cases} \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We showed that $\lim_{x\to 0} f(x)$ does not exist. So, f is discontinuous at $x = 0$. We will show later on that f is continuous at any point $a \neq 0$.

Verifying directly that a given function is continuous can be tedious. Thankfully, we can often verify continuity of a function using the following rules, which are consequence of the Limit Laws, Proposition 1.16.

Proposition 1.28 (**Laws of Continuity**)**.** *Let $f, g\colon \mathbb{R} \to \mathbb{R}$. Let $a, c \in \mathbb{R}$. Assume that f and g are both continuous at a. Then*

(i) *$f + g$ is continuous at a.*
(i') *$f - g$ is continuous at a*
(ii) *cf is continuous at a.*
(iii) *fg is continuous at a*
(iv) *If $g(a) \neq 0$, then f/g is continuous at a.*

Corollary 1.29. *Let p, q be polynomials. Then p is continuous on the real line, and p/q is continuous at all values a where $q(a) \neq 0$.*

Proof. Note that $p(x)$ is a sum of monomials $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. Each monomial is continuous, so p is continuous by the first law of continuity.

So, p, q are both continuous on all of $\mathbb{R}$. Then p/q is continuous at all values a where $q(a) \neq 0$, by the last law of continuity. □

Proposition 1.30.

- *$f(x) = \sin x$ and $g(x) = \cos x$ are both continuous on the real line.*
- *If $b > 0$, then $f(x) = b^x$ is continuous on the real line.*
- *If $b > 0$, then $f(x) = \log_b x$ is continuous whenever $x > 0$.*
- *If n is a positive integer, then $f(x) = x^{1/n}$ is continuous on its domain.*

Example 1.31. Recall that $\cos(x) = 0$ whenever $x = k\pi + \pi/2$ for some integer k. So, $\tan(x) = \sin(x)/\cos(x)$ is continuous, except when $x = k\pi + \pi/2$ for some integer k.

Example 1.32. The function $f(x) = x^2 \cos x$ is continuous on the real line, by the third law of continuity.

Theorem 1.33 (**A Composition of Continuous Functions is Continuous**). *Let f, g be functions. Define $F(x) = f(g(x))$. If g is continuous at $x = a$, and if f is continuous at $g(a)$, then $F(x) = f(g(x))$ is continuous at $x = a$.*

Example 1.34. Recall the function

$$F(x) = \begin{cases} \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Since $g(x) = 1/x$ is continuous for $x \neq 0$, and $f(y) = \cos(y)$ is continuous for any y, we see that $F(x) = f(g(x))$ is continuous for any $x \neq 0$. And we saw before that F is discontinuous at $x = 0$.

Remark 1.35. If $f \colon \mathbb{R} \to \mathbb{R}$ is continuous, we can move limits inside or outside of f:

$$\lim_{x\to a} f(x) = f(\lim_{x\to a} x).$$

Example 1.36. $\lim_{x\to 1} \sqrt{x^2+1} = \sqrt{\lim_{x\to 1}(x^2+1)} = \sqrt{2}$.

Example 1.37. $\lim_{x\to\pi/4} \sin(2x - \pi/4) = \sin(\lim_{x\to\pi/4}(2x-\pi/4)) = \sin(\pi/4) = \sqrt{2}/2$.

Sometimes we need to do a bit of algebra before we can move the limits around or apply limits laws.

Example 1.38.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x\to 2}(x+2) = 4.$$

Example 1.39.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}.$$

Example 1.40.

$$\lim_{x\to 0} \left(\frac{1}{4x} - \frac{1}{x(x+4)}\right) = \lim_{x\to 0} \frac{x+4-4}{4x(x+4)} = \lim_{x\to 0} \frac{x}{4x(x+4)} = \lim_{x\to 0} \frac{1}{4(x+4)} = \frac{1}{16}.$$

1.4. **Trigonometric Limits, Limits at Infinity.**

Theorem 1.41 (**Squeeze Theorem**). *Let $f, g, h \colon \mathbb{R} \to \mathbb{R}$. Suppose $f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} g(x) = L$.*

Example 1.42. We show that $\lim_{x\to 0} x\cos(x) = 0$. Since $-1 \leq \cos(x) \leq 1$, we have $-|x| \leq x\cos x \leq |x|$. Since $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$, we conclude that $\lim_{x\to 0} x\cos x = 0$.

Corollary 1.43.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

Proof. If x is small, then the right triangle with vertices $(0,0)$, $(\cos x, 0)$ and $(\cos x, \sin x)$ is inscribed in the unit circle. The height of the triangle is larger than the arc of the circle between $(\cos x, \sin x)$ and $(1, 0)$. This arc has length x. So, $\sin(x) \leq x$. On the other hand, the sector formed by this arc of the circle has area $(1/2)x$. And this sector is contained in the right triangle with one edge formed by $(0,0)$ and $(1,0)$ and the other edge with slope

$\tan x$. This triangle has area $(1/2)\sin(x)/\cos(x)$. So, $(1/2)x \leq (1/2)\sin(x)/\cos(x)$, i.e. $\cos(x) \leq \sin(x)/x$. In summary,

$$\cos(x) \leq \frac{\sin x}{x} \leq 1.$$

Since $\lim_{x\to 0}\cos(x) = \cos(0) = 1$, and $\lim_{x\to 0} 1 = 1$, the Squeeze Theorem implies that $\lim_{x\to 0}(\sin x)/x = 1$. Lastly,

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x}\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$
$$= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right) = 1\cdot 0 = 0.$$

□

1.4.1. *Limits at Infinity.* So far we have only considered limits of the form $\lim_{x\to a} f(x)$ where a is a real number. We can also consider limits of the form $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, using essentially the same rules as before. Alternatively, we could make the following definition, if the limit on the right exists:

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0^+} f(1/y).$$

Similarly, if the limit on the right exists, we can define

$$\lim_{x\to-\infty} f(x) = \lim_{y\to 0^+} f(-1/y).$$

We can think of $\lim_{t\to\infty} f(t)$ as describing where f eventually ends up, as time goes to infinity.

Example 1.44. $\lim_{x\to\infty} 1/x = 0$.

Example 1.45. $\lim_{x\to\infty} 2^x = \infty$, $\lim_{x\to\infty} 2^{-x} = 0$.

Example 1.46.

$$\lim_{x\to\infty}\frac{4x^2+2}{3x^2+x+3} = \lim_{x\to\infty}\frac{4x^2+2}{3x^2+x+3}\cdot\frac{x^{-2}}{x^{-2}} = \lim_{x\to\infty}\frac{4+\frac{2}{x^2}}{3+\frac{1}{x}+\frac{3}{x^2}} = \frac{\lim_{x\to\infty} 4+\frac{2}{x^2}}{\lim_{x\to\infty} 3+\frac{1}{x}+\frac{3}{x^2}} = \frac{4}{3}.$$

Example 1.47. More generally, if n is a positive integer and $d_n \neq 0$,

$$\lim_{x\to\infty}\frac{c_n x^n + c_{n-1}x^{n-1}+\cdots+c_0}{d_n x^n + d_{n-1}x^{n-1}+\cdots+d_0} = \lim_{x\to\infty}\frac{c_n + \frac{c_{n-1}}{x}+\cdots+\frac{c_0}{x^n}}{d_n + \frac{d_{n-1}}{x}+\cdots+\frac{d_0}{x^n}}$$
$$= \frac{\lim_{x\to\infty} c_n + \frac{c_{n-1}}{x}+\cdots+\frac{c_0}{x^n}}{\lim_{x\to\infty} d_n + \frac{d_{n-1}}{x}+\cdots+\frac{d_0}{x^n}} = \frac{c_n}{d_n}.$$

1.5. **Intermediate Value Theorem.** Continuous functions have many nice properties. One such property is expressed in the following Theorem.

Theorem 1.48 (**Intermediate Value Theorem**). *Let $a,b\in\mathbb{R}$, $a<b$. Let $f\colon[a,b]\to\mathbb{R}$ be continuous. Then f achieves every value between $f(a)$ and $f(b)$. That is, for every $y\in\mathbb{R}$ in between $f(a)$ and $f(b)$, there exists $x\in[a,b]$ such that $f(x)=y$.*

Example 1.49. For a real-world example, suppose $f\colon [0,10] \to \mathbb{R}$ and $f(t)$ is your position at time t as you walk along a straight path. Suppose $f(0) = 0$ and $f(10) = 2$. Then you must visit every point along the path $[0,2]$ at some time between time 0 and time 10.

Example 1.50. There is some nonzero number x such that $2\sin(x) = x$. To see this, let $g(x) = 2\sin(x) - x$. Then $g(\pi/2) = 2 - \pi/2 > 0$ and $g(\pi) = -\pi < 0$. By the Intermediate Value Theorem, there is some $x \in (\pi/2, \pi)$ such that $g(x) = 0$. At this point, we have $2\sin(x) = x$.

Example 1.51. Continuity is needed for the conclusion of the Intermediate Value Theorem to hold. For example, consider the jump discontinuity

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Then $H(-1) = 0$, $H(1) = 1$, but there does not exist some $x \in (-1,1)$ with $H(x) = 1/2$.

2. The Derivative

2.1. **Definition of the Derivative.**

The derivative is one of the central concepts in calculus. It is also one of the most useful. For a function $f\colon \mathbb{R} \to \mathbb{R}$, where $f(t)$ denotes the vertical position of a falling object at time t, the derivative of t is the velocity of the object at time t. For a general function $f\colon \mathbb{R} \to \mathbb{R}$, the derivative of f is the rate of change of f. We denote the derivative of f at x by $f'(x)$.

We now begin the construction of the derivative. Let $f\colon \mathbb{R} \to \mathbb{R}$. Suppose Let $a, x \in \mathbb{R}$ with $a < t$. Consider the points $(a, f(a))$ and $(x, f(x))$. Recall that the line passing through these points has a slope equal to the **rise over run** of the points. That is, the slope of the line passing through these two points is

$$\frac{f(x) - f(a)}{x - a}.$$

The numerator is the change in the value of f from a to x, and the denominator is the change in the domain of f from a to x. So, $(f(x) - f(a))/(x - a)$ is essentially the rate of change of f at a. The line passing through the points $(a, f(a))$ and $(x, f(x))$ is also known as the **secant line**.

The slope of the secant line is a decent approximation to the slope of the tangent line to f at a. It turns out that as $x \to a$, the slope of the secant line will often go to the slope of the tangent line. That is, suppose as x goes to a, the quantity $\frac{f(x)-f(a)}{x-a}$ goes to some number L. We can then think of L as the infinitesimal rate of change of f. To see this approximation procedure in action, see the following JAVA applet: secant approximation.

This discussion leads the following definition of a derivative

Definition 2.1 (**The Derivative at a Point**)**.** Let $a, b, x \in \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$. We say that f is **differentiable** at a if the following limit exists.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) = \frac{df(a)}{dx} = \frac{df}{dx}(a) = \frac{d}{dx} f(a).$$

If $f'(a)$ exists, we call $f'(a)$ the **derivative** of f at a. We say that f is **differentiable** if f is differentiable on its domain.

Example 2.2. Let $f(x) = c$ be a constant function. Then $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0$.

Let $f(x) = x$. Then $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$.

Let $f(x) = x^2$. Then $f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{1+2h+h^2-1}{h} = \lim_{h\to 0}(2+h) = 2$.

Let $f(x) = x^{-1}$. Then

$$f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{3-(3+h)}{3(3+h)h} = \lim_{h\to 0}\frac{-1}{3(3+h)} = -\frac{1}{9}.$$

Definition 2.3 (Tangent Line). Let $f\colon \mathbb{R}\to\mathbb{R}$. Assume f is differentiable at a. The **tangent line** to the graph $y = f(x)$ at the point $x = a$ is the line with slope $f'(a)$ which passes through the point $(a, f(a))$. So, this line has equation

$$y = f(a) + (x-a)f'(a).$$

Example 2.4. Let's find the tangent line to the curve $y = x^2$ at $x = 1$. If $f(x) = x^2$, then $f(1) = 1$, and we computed already $f'(1) = 2$. So, the tangent line is $y = 1 + 2(x-1)$.

Example 2.5. Let's find the tangent line to the curve $y = x^{-1}$ at $x = 3$. If $f(x) = x^{-1}$, then $f(3) = 1/3$, and we computed already $f'(3) = -1/9$. So, the tangent line is $y = (1/3) - (1/9)(x-3)$.

Recall that if f is a constant function, we computed $f'(0) = 0$. And if $f(x) = x$, we computed $f'(0) = 1$. Imitating these calculations leads to the following proposition.

Proposition 2.6.

- *Suppose $f(x) = c$ where c is a constant. Then $f'(a) = 0$ for all a.*
- *Suppose $f(x) = mx + b$ where m, b are constants. Then $f'(a) = m$ for all a.*

2.2. **The Derivative as a Function.** Let $x \in \mathbb{R}$, and let $f\colon \mathbb{R}\to\mathbb{R}$. If $f'(x)$ exists, then

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) = \frac{df}{dx} = \frac{df}{dx}(x) = \frac{d}{dx}f(x)$$

Note that $f'(x)$ is also a function of x.

Example 2.7. Let $f(x) = x^2$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}(2x+h) = 2x. \end{aligned}$$

So, $f'(x) = 2x$.

We can generalize this computation to arbitrary powers of positive integers.

Proposition 2.8. *Let $f(x) = x^n$, where n is any constant. Then $f'(x) = nx^{n-1}$. (If $n < 1$ and $n \neq 0$, then $f'(0)$ is undefined.)*

Proof. For simplicity, we assume n is a positive integer. Let $a, b > 0$. We need a generalization of the identity $a^2 - b^2 = (a-b)(a+b)$, or $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).$$

Using this identity with $a = (x+h)$ and $b = x$, we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{h}{h}((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}) \\ &= \lim_{h\to 0}((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}) \\ &= \underbrace{x^{n-1} + \cdots + x^{n-1}}_{n \text{ times}} = nx^{n-1}. \end{aligned}$$

□

Example 2.9. If $f(x) = x^3$, then $f'(x) = 3x^2$.
If $f(x) = x^{2/3}$, then $f'(x) = (2/3)x^{-1/3}$.
If $f(x) = x^{-\sqrt{2}}$, then $f'(x) = -\sqrt{2}x^{-\sqrt{2}-1}$.

Remark 2.10. There is a real number denoted by $e \approx 2.718...$ such that $\frac{d}{dx}e^x = e^x$.

Proposition 2.11. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$. Since f is differentiable at a, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. For $x \neq a$, write $f(x) - f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$. Then our limit law for products applies, yielding

$$\begin{aligned} \lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right) \\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right). = f'(x)\cdot 0 = 0 \end{aligned}$$

So, $\lim_{x\to a} f(x) = f(a)$, i.e. f is continuous at a. □

Is there a continuous function that is not differentiable at one point? In other words, does the converse to Proposition 2.11 hold?

Example 2.12 (**A Discontinuous Derivative**)**.** Consider the function $f(x) = |x|$, $x \in \mathbb{R}$. Since $f(x) = x$ for $x > 0$, and $f(x) = -x$ for $x < 0$, we see that $f'(x)$ resembles the Heaviside function

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

But what happens at zero? Observe,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1,$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} (-1) = -1.$$

So, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist, i.e. $f'(0)$ does not exist and the converse of Proposition 2.11 is false. In fact, a stronger statement holds.

From the contrapositive of Proposition 2.11, we know that if f is discontinuous, then f is not differentiable. There are even more ways for f to not be differentiable.

Example 2.13 (**A Derivative Approaching Infinity**). Let $x \in \mathbb{R}$, and let $f(x) = x^{1/3}$. For $x \neq 0$, $f'(x) = (1/3)x^{-2/3}$. So, $\lim_{x\to 0} f'(x) = \infty$, i.e. $\lim_{x\to 0} f'(x)$ does not exist. Also, $f'(0)$ does not exist.

2.3. **Product Rule, Quotient Rule, Chain Rule.**

Proposition 2.14 (**Properties of the Derivative**). *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions. Let $c \in \mathbb{R}$ be a constant.*

- $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$.
- $\frac{d}{dx}(f(x) + g(x)) = \left(\frac{d}{dx}f(x)\right) + \left(\frac{d}{dx}g(x)\right)$
- $\frac{d}{dx}\left(f(x) - g(x)\right) = \left(\frac{d}{dx}f(x)\right) - \left(\frac{d}{dx}g(x)\right)$.
- *(**Product rule**)* $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$.
- *(**Quotient rule**) If $g(x) \neq 0$, then*
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$
- *(**Chain rule**) If $f, g\colon \mathbb{R} \to \mathbb{R}$ are differentiable, then*
$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$
Writing $u = g(x)$, and $y = f(g(x))$, we can also write this as
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Remark 2.15. The quotient rule can be memorized with the mnemonic "low d-hi minus hi d-low over low squared."

Example 2.16.
$$\frac{d}{dx}(x^4 + 3x^2 + 2x^{1/2} + 1) = (\frac{d}{dx}x^4) + 3(\frac{d}{dx}x^2) + 2(\frac{d}{dx}x^{1/2}) + (\frac{d}{dx}1) = 4x^3 + 6x + x^{-1/2}.$$

Example 2.17. Let $f(x) = x^2 - 2x + 1 = (x-1)^2$. Note that $f'(x) = 2x - 2 = 2(x-1)$. So, when $x > 1$, $f'(x) > 0$, and f is increasing. And when $x < 1$, $f'(x) < 0$, and f is decreasing. Finally, when $x = 1$, $f'(x) = 0$, so f is neither increasing nor decreasing, and the tangent line to f is horizontal.

Example 2.18. Let $f(x) = x^3 + x$ and let $g(x) = x^{1/2} - 3$. Then $(d/dx)(f(x)g(x)) = (d/dx)[(x^3 + x)(x^{1/2} - 3)] = (x^3 + x)(x^{-1/2}/2) + (x^{1/2} - 3)(3x^2 + 1)$.

Example 2.19. Let $f(x) = x^3$ and let $g(x) = x^{1/2} - 3$. Then $(d/dx)(f(g(x))) = (d/dx)(x^{1/2} - 3)^3 = 3(x^{1/2} - 3)^2(1/2)x^{-1/2}$.

Example 2.20. Let $f(x) = \frac{x}{x^2+1}$. Then $f'(x) = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$.

Example 2.21. Let $f(x) = \frac{1}{x^3+2}$. Then $f'(x) = \frac{(x^3+2)\frac{d}{dx}(1) - 1\frac{d}{dx}(x^3+2)}{(x^3+2)^2} = \frac{-3x^2}{(x^3+2)^2}$.

Proof of the Product Rule. Let $x, h \in \mathbb{R}$. Then

$$\begin{aligned}
&\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&\quad = \frac{f(x+h)g(x+h) - f(x)g(x+h) + g(x+h)f(x) - f(x)g(x)}{h} \\
&\quad = \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{g(x+h)f(x) - f(x)g(x)}{h} \\
&\quad = g(x+h)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

Since g is differentiable, g is continuous by Proposition 2.11, so that $\lim_{h\to 0} g(x+h) = g(x)$. Applying our limit laws (which ones, and how are they justified?), we get

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&\quad = \left(\lim_{h\to 0} g(x+h)\frac{f(x+h) - f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\frac{g(x+h) - g(x)}{h}\right) \\
&\quad = g(x)f'(x) + f(x)g'(x).
\end{aligned}$$

□

Proof of the Quotient Rule. Let x such that $g(x) \neq 0$. We first show that $(1/g(x))$ is differentiable, and

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{(g(x))^2}. \qquad (*)$$

Observe,

$$\begin{aligned}
\frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right) &= \frac{1}{h}\left(\frac{g(x)}{g(x)g(x+h)} - \frac{g(x+h)}{g(x)g(x+h)}\right) \\
&= \frac{g(x) - g(x+h)}{h}\frac{1}{g(x)g(x+h)}. \qquad (**)
\end{aligned}$$

Since g is differentiable, g is continuous by Proposition 2.11, so $\lim_{h\to 0} g(x+h) = g(x)$. Using our quotient limit law,

$$\lim_{h\to 0} \frac{1}{g(x+h)} = \frac{1}{\lim_{h\to 0} g(x+h)} = \frac{1}{g(x)}.$$

So, taking the limit of $(**)$, and using our limit law for products,

$$\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right) = \left(\lim_{h\to 0} \frac{g(x) - g(x+h)}{h}\right)\left(\lim_{h\to 0} \frac{1}{g(x)g(x+h)}\right) = \frac{g'(x)}{(g(x))^2}.$$

Since $\lim_{h\to 0} \frac{1}{h}\left(\frac{1}{g(x+h)} - \frac{1}{g(x)}\right)$ exists, $1/g(x)$ is differentiable, and formula $(*)$ is proven. We can now conclude by using the product rule. Observe

$$\begin{aligned}
\frac{d}{dx}\left(f(x)\frac{1}{g(x)}\right) &= f(x)\frac{d}{dx}\left(\frac{1}{g(x)}\right) + f'(x)\frac{1}{g(x)} \\
&= -f(x)\frac{g'(x)}{(g(x))^2} + f'(x)\frac{g(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

□

Proof sketch of the Chain Rule.

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\ &= \left(\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\\ &= \left(\lim_{y\to g(x)}\frac{f(y)-f(g(x))}{y-g(x)}\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right) = f'(g(x))g'(x).\end{aligned}$$

□

Corollary 2.22. *Let $f,g\colon\mathbb{R}\to\mathbb{R}$ be differentiable. Let n be a positive integer, let m,b be constants. Then*

- $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x)$.
- $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$.
- $\frac{d}{dx}f(mx+b) = mf'(mx+b)$.

2.4. **Higher Derivatives.** Let $f\colon\mathbb{R}\to\mathbb{R}$. Recall that f' is also a function, and $f'>0$ when f is increasing, while $f'<0$ when f is decreasing. Also, if $f(t)$ represents the position of an object at time t, then $f'(t)$ is the velocity of the object at time t, since $f'(t)$ is the rate of change of the position over time. We can also consider the rate of change of the velocity over time, which is known as acceleration. That is, we can consider the derivative $(d/dt)f'(t)$ of $f'(t)$ to be the acceleration of the object at time t. We denote this second derivative as $f''(t)$.

Definition 2.23 (Higher Derivatives). Let $f\colon\mathbb{R}\to\mathbb{R}$. We define the **second derivative** of f at the point $x\in\mathbb{R}$ by

$$f''(x) = (d/dx)f'(x) = \frac{d^2}{dx^2}f(x).$$

We similarly define the third derivative of f by

$$f'''(x) = (d/dx)f''(x) = \frac{d^3}{dx^3}f(x),$$

we define the fourth derivative of f by $f^{(4)}(x) = (d/dx)f'''(x) = d^4f(x)/dx^4$, and so on. In general, for any positive integer n, we define the n^{th} derivative of f by $f^{(n)}(x) = (d/dx)f^{(n-1)}(x) = \frac{d^n}{dx^n}f(x)$.

Example 2.24. Let $f(x) = x^3-3x+1$. Then $f'(x) = 3x^2-3$, $f''(x) = 6x$, $f'''(x) = 6$, $f^{(4)}(x) = 0$, $f^{(5)}(x) = 0$, and all higher derivatives of f are zero.

Example 2.25. Suppose you throw a baseball vertically in the air, with initial upward velocity v_0 and initial position r_0 (and we neglect air friction). Then, the position of the baseball (in meters) at time t (in seconds) is

$$r(t) = r_0 + tv_0 - (9.8/2)t^2.$$

Note that $r(0) = r_0$, $r'(0) = v_0$, and $r''(t) = -9.8$. That is, for any time t, the acceleration of the baseball (due to gravity) is constant. Also, all higher derivatives of r are zero: $r'''(t) = 0$, $r^{(4)}(t) = 0$, and so on.

2.5. **Trigonometric Functions.**

- $\frac{d}{dx}(\sin(x)) = \cos(x)$, $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- $\frac{d}{dx}(\tan(x)) = (\sec(x))^2$, $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
- $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$, $\frac{d}{dx}(\cot(x)) = -(\csc(x))^2$.

Proof. We first recall that $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$. Recall also that $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$. So, applying our limit law for addition, we get

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \sin(x)\left(\lim_{h\to 0}\frac{(\cos(h)-1)}{h}\right) + \cos(x)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)\\ &= (\sin(x))\cdot 0 + (\cos(x))\cdot 1 = \cos(x).\end{aligned}$$

So, $(d/dx)\sin x = \cos(x)$. Then, since $\cos(x) = \sin(x+\pi/2)$, we have $(d/dx)\cos(x) = (d/dx)\sin(x+\pi/2) = \cos(x+\pi/2) = -\sin(x)$. The remaining identities follow from the first two. □

Example 2.26.

$$\begin{aligned}\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$

2.6. **Implicit Differentiation.** Suppose we have a curve satisfying some equation in x and y. For example, consider the unit circle

$$x^2 + y^2 = 1.$$

We would like to find dy/dx, but maybe it is difficult or impossible to solve for y directly. In such a case, we can use implicit differentiation. That is, we treat y as a function of x, so that $y = y(x)$. And we simply differentiate the equation $x^2 + y^2 = 1$ with respect to x. For example, we get

$$2x + 2y(dy/dx) = 0.$$

So, if $y \neq 0$, we solve for dy/dx to get

$$\frac{dy}{dx} = -\frac{x}{y}.$$

For example, at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, the slope of the tangent line is $dy/dx = -\frac{\sqrt{2}}{\sqrt{2}} = -1$.

Example 2.27. Consider the equation $x^3 + y\sin(x) + y^3 = 1$. We want to find dy/dx. We write $y = y(x)$ and differentiate with respect to x to get

$$3x^2 + \sin(x)(dy/dx) + y\cos(x) + 3y^2(dy/dx) = 0.$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = -\frac{y\cos x + 3x^2}{\sin x + 3y^2}.$$

Example 2.28. Let $x^2 + y^2 = 1$. We showed that $y'(x) = -x/y$, or $yy' = -x$. We can differentiate implicitly again to obtain d^2y/dx^2. Implicitly differentiating, we have $yy'' + (y')^2 = -1$. So,

$$y''(x) = \frac{-1 - (y')^2}{y} = \frac{-1 - x^2/y^2}{y} = \frac{-y^2 - x^2}{y^3} = -\frac{1}{y^3}.$$

2.7. **Related Rates.** In a related rates problem, typically some quantity is changing, and we would like to find the rate of change of another related quantity. These problems are best understood with examples.

Example 2.29. Suppose the radius of a circle is increasing at a rate of 1 meter per second. What is the rate of change of the area of the circle, when the radius is 3 meters?

Let r be the radius of the circle. Then the area satisfies $A = A(r) = \pi r^2$. So, considering r to be a function of t, we have $A(r(t)) = \pi r(t)^2$. So, $dA/dt = 2\pi r(dr/dt) = 2\pi r$, since $dr/dt = 1$. When $r = 3$, we therefore have $dA/dt = 6\pi$. Alternatively, we could have used the Chain Rule in the form $dA/dt = (dA/dr)(dr/dt) = 2\pi r(dr/dt)$.

Example 2.30. Suppose a vertical cylindrical tank of radius 3 meters is draining water at a rate of 3000 Liters per minute. How fast is the level of water dropping?

Let h be the height of water in the tank. The current volume of water in the tank is $V = \pi r^2 h = 9\pi h$ cubic meters. Recall that 1000 Liters is one cubic meter. Considering h to be a function of t, we have $dV/dt = (dV/dh)(dh/dt) = 9\pi(dh/dt)$. We know that $dV/dt = -3$ cubic meters per minute. So, $dh/dt = -3/(9\pi) = -1/(3\pi)$ meters per minute.

Example 2.31. A 10 foot ladder is leaning against a wall. While touching both the wall and the ground, the bottom of the ladder is sliding away from the wall at a rate of 1 foot per second. What is the speed of the top of the ladder when the top is 6 feet off of the ground?

Let h be distance of the top of the ladder from the ground, and let r be the length from the bottom of the ladder to the wall. The ladder makes a right triangle with legs r, h and hypotenuse 10. So, from the Pythagorean Theorem, $r^2 + h^2 = 10^2 = 100$. Differentiating this equation with respect to t, we have $2r(dr/dt) + 2h(dh/dt) = 0$. When $h = 6$, we have $r = 8$ since $r^2 + h^2 = 100$. Also, we know that $dr/dt = 1$. So, we have $2(8)(1) + 2(6)(dh/dt) = 0$. So, $dh/dt = -8/6 = -4/3$ feet per second.

3. Applications of Derivatives

3.1. **Linear Approximation.** When we look at a very small domain of a function f near a point a, the function f looks like a linear function. This heuristic is expressed with the following approximation

$$f(x) \approx f(a) + (x - a)f'(a).$$

That is, when x is near a, the quantity $f(x)$ is approximately equal to the quantity $f(a) + (x - a)f'(a)$.

Definition 3.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ and let a be a point in $\mathbb{R}$. The **linearization** of f near a is the following function of x:

$$L(x) = f(a) + (x - a)f'(a).$$

The expression

$$f(x) \approx L(x)$$

is referred to as the **linear approximation** of f at a. Note that $y = L(x)$ is exactly the equation for the tangent line to f at a.

Example 3.2. Let's estimate $\sin(\pi/6 + .01)$ using linear approximations. We approximate $\sin(x)$ by $L(x)$ when $a = \pi/6$. We have

$$L(x) = \sin(\pi/6) + (x - \pi/6)\sin'(\pi/6) = (1/2) + (x - \pi/6)\cos(\pi/6) = (1/2) + (x - \pi/6)\sqrt{3}/2.$$

So, using the linear approximation of $\sin(x)$ at $\pi/6$, we have

$$\sin(\pi/6 + .01) \approx L(\pi/6 + .01) = (1/2) + (.01)\sqrt{3}/2.$$

Example 3.3. Let's estimate $1/(9.9)$ using linear approximation. We approximate $1/x$ by $L(x)$ when $a = 10$. We have

$$L(x) = 1/10 + (x - 10)(-(10)^{-2}) = 1/10 - (x - 10)/100.$$

So, we have

$$1/(9.9) \approx L(9.9) = 1/10 - (-.1)/100 = 1/10 + 1/1000 = \frac{101}{1000}.$$

3.2. **Extreme Values and Optimization.**

Nothing takes place in the world whose meaning is not that of some maximum or minimum.

Leonhard Euler

Given a function f, it is often desirable to find the maximum or minimum value of f. For example, if $f(x)$ represents the profit obtained from setting the price x of a product, we would like to maximize the profit function f. If f represents the energy or cost of an object moving between two points, we would like to minimize this energy or cost. Many physical principles can be rephrased as statements about minimizing energy or some other quantity. For example, light always travels on the path of shortest length between two points. That is, photons minimize the length over which they travel. And so on. Thankfully there are often very general methods for maximizing or minimizing functions. Unfortunately, these general methods do not always work. For example, suppose the mailman has 1000 houses to visit, and he wants to visit them in the shortest amount of time. We do not yet know an efficient way to find the path that takes the shortest amount of time.

Definition 3.4 (**Extrema**). Let D be a domain in the real line. Let $f\colon D \to \mathbb{R}$. We say that f has an **absolute maximum** on D at the point $c \in D$ if

$$f(x) \leq f(c) \qquad \text{for all}\, x \in D.$$

We say that f has an **absolute minimum** on D at the point $c \in D$ if

$$f(x) \geq f(c) \qquad \text{for all}\, x \in D.$$

We refer to the absolute minimum and absolute maximum values of f as the **extreme values** or **extrema** of f. The process of finding the extrema of f is called **optimization**.

Since we would often like to find the extrema of a function f, let's first look at some examples where the extrema cannot possibly be found (since they don't exist).

Example 3.5. Let $f(x) = 1/x$ where f has domain $(0, 1)$. Then f has no absolute maximum since $\lim_{x\to 0^+} f(x) = \infty$. So, a discontinuity of a function can interfere with the existence of extrema.

Example 3.6. Let $f(x) = x^2$ where f has domain $[0, 2]$. Then the absolute maximum of f is 4, which occurs at $x = 2$. And the absolutely minimum of f is 0, which occurs at $x = 0$.

Let $f(x) = x^2$ where f has domain $(0, 2)$. Then the absolute maximum of f does not exist. And the absolutely minimum of f does not exist either. Given any point $c \in (0, 2)$, there are always points $a, b \in (0, 2)$ with $a < c < b$, so that $f(a) < f(c) < f(b)$. So, an extreme value cannot occur at any point in $(0, 2)$. Even though f is continuous, the open interval is interfering with the existence of extrema.

In summary, if we want extrema to exist, it looks like we need our function to be continuous, and we cannot consider a domain which is an open interval. Fortunately, being continuous on a closed interval guarantees that the extrema exist.

Theorem 3.7 (Extreme Value Theorem). *Let $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f achieves its minimum and maximum values. More specifically, there exist $c, d \in [a, b]$ such that: for all $x \in [a, b]$, $f(c) \leq f(x) \leq f(d)$.*

The proof of this Theorem is outside the scope of this course; at UCLA it would be proven in Math 131A.

Sometimes it is also desirable to find extreme values for a function when it is restricted to a small domain. Such points are called **local extrema**.

Definition 3.8 (Local Extrema). Let $f : \mathbb{R} \to \mathbb{R}$. We say that f has an **local maximum** at the point $c \in \mathbb{R}$ if

$$f(x) \leq f(c) \qquad \text{for all points } x \text{ in some open interval containing } c.$$

We say that f has an **local minimum** at the point $c \in \mathbb{R}$ if

$$f(x) \geq f(c) \qquad \text{for all points } x \text{ in some open interval containing } c.$$

Finding local extrema often reduces to the problem of finding critical points.

Definition 3.9 (Critical Point). Let $f : \mathbb{R} \to \mathbb{R}$. We say that f has a **critical point** at the point c if $f'(c) = 0$ or $f'(c)$ does not exist.

Example 3.10. Let $f(x) = x^2$. Then $f'(x) = 2x$. So, $f'(x) = 0$ only when $x = 0$. That is, $x = 0$ is the only critical point of f. Note that $x = 0$ is also the absolute minimum for f with the domain $\mathbb{R}$.

Example 3.11. Let $f(x) = |x|$. If $x < 0$, then $f'(x) = -1$. If $x > 0$, then $f'(x) = 1$. If $x = 0$, then $f'(x)$ is undefined. So, $x = 0$ is the only critical point of f. Note that $x = 0$ is also the absolute minimum for f with the domain $\mathbb{R}$.

Example 3.12. Let $f(x) = x^3$. Then $f'(x) = 3x^2$. So, $f'(x) = 0$ only when $x = 0$. That is, $x = 0$ is the only critical point of f. Note that $x = 0$ is neither a local maximum nor a local minimum of f. So, a critical point is not necessarily a local extremum.

We have just seen that a critical point is not necessarily a local extremum. However, a local extremum is always a critical point.

Theorem 3.13. *Let $f : \mathbb{R} \to \mathbb{R}$. If a local minimum or maximum of f occurs at c, then c is a critical point of f.*

Proof. Suppose c is a local maximum of f and $f'(c)$ exists. Then c is an absolute maximum of f on some open interval $(c-b, c+b)$ where $b > 0$. Let $h \in (-b, b)$. Then $f(x+h) \leq f(c)$, since c is a local maximum of f. If $h > 0$, we have $(f(x+h) - f(c))/h \geq 0$. So,

$$\lim_{h\to 0^+} \frac{f(x+h) - f(c)}{h} \geq 0.$$

If $h < 0$, we have $(f(x+h) - f(c))/h \leq 0$. So,

$$\lim_{h\to 0^-} \frac{f(x+h) - f(c)}{h} \leq 0.$$

We therefore have $0 \leq f'(c) \leq 0$, so that $f'(c) = 0$ (if $f'(c)$ exists). □

This Theorem allows us to find absolute extrema on closed intervals.

Proposition 3.14 (**Extreme Values on Closed Intervals**). *Let $a < b$. Let $f: [a, b] \to \mathbb{R}$ be continuous. Then the extreme values of f on $[a, b]$ occur either at critical points of f, or at the endpoints a, b of the interval $[a, b]$.*

Proof. Let c be a point where an extreme value of f occurs, and assume $c \neq a$ and $c \neq b$. Then c is a local extremum of f on (a, b). So, c is a critical point of f by the Theorem 3.13. □

Example 3.15. Let's find the extreme values of $f(x) = x^2 - 3x + 1$ on the interval $[0, 2]$. We have $f'(x) = 2x - 3$, so that $f'(x) = 0$ only when $x = 3/2$. So, the only critical point of f occurs at $x = 3/2$. So, the extreme values of f must occur at elements of the following list: $0, 3/2, 2$. We now check the values of f at these points. We have $f(0) = 1$, $f(3/2) = (9/4) - (9/2) + 1 = -5/4$, and $f(2) = 4 - 6 + 1 = -1$.

So the absolute maximum of f on $[0, 2]$ is 1, which occurs at $x = 0$. And the absolute minimum of f on $[0, 2]$ is $-5/4$ which occurs at $x = 3/2$.

Example 3.16. Let's find the extreme values of $f(x) = x - \sin(x)$ on the interval $[-\pi/2, \pi/2]$. We have $f'(x) = 1 - \cos(x)$, so that $f'(x) = 0$ only when $\cos(x) = 1$. When $x \in [-\pi/2, \pi/2]$, we only have $\cos(x) = 1$ at $x = 0$. So, $x = 0$ is the only critical point of f. So, the extreme values of f must must occur at elements of the following list: $-\pi/2, 0, \pi/2$. We now check the values of f at these points. We have $f(0) = 0$, $f(\pi/2) = \pi/2 - 1$, and $f(-\pi/2) = -\pi/2 + 1$.

So the absolute maximum of f on $[-\pi/2, \pi/2]$ is $\pi/2 - 1$, which occurs at $x = \pi/2$. And the absolute minimum of f on $[-\pi/2\pi/1]$ is $-\pi/2 + 1$ which occurs at $x = -\pi/2$.

3.3. **Mean Value Theorem.** The following theorems allow us to better understand the graphs of functions.

Theorem 3.17 (**Rolle's Theorem**). *Let $f: [a, b] \to \mathbb{R}$ be continuous function that is differentiable on (a, b) with $f(a) = f(b) = 0$. Then there exists c with $c \in (a, b)$ and $f'(c) = 0$.*

Proof. From the Extreme Value Theorem, let c be an extreme value of f on $[a, b]$. If an extreme value c occurs in (a, b), then $f'(c) = 0$ by Theorem 3.13. If not, both the max and min occur at the endpoints a, b, which implies that f is a constant function. But if f is a constant, then $f' = 0$ everywhere. So, in any case, there is a $c \in (a, b)$ with $f'(c) = 0$. □

Theorem 3.18. ***(Mean Value Theorem)*** *Let $f: [a, b] \to \mathbb{R}$ be continuous function that is differentiable on (a, b). Then there exists c with $c \in (a, b)$ and*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $g(x) = f(x) - f(a) - (x - a)((f(b) - f(a))/(b - a))$. Then $g(a) = g(b) = 0$. So, by applying Rolle's Theorem, Theorem 3.17, to g, there exists $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

That is, $f'(c) = (f(b) - f(a))/(b - a)$. □

Remark 3.19. Rolle's Theorem corresponds to the case $f(b) = f(a)$ in the Mean Value Theorem.

Example 3.20. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(1) = 1$. Then there is some point $x \in (0, 1)$ such that $f'(x) = (f(1) - f(0))/(1 - 0) = 1$.

Recall that a constant function has a derivative that is zero. The converse is also true, though we technically did not know it to be true until now.

Corollary 3.21 (**Functions with Zero Derivative are Constant**)**.** *Let $a < b$. Let $f: (a, b) \to \mathbb{R}$ be differentiable with $f'(x) = 0$ for all $x \in (a, b)$. Then there is a constant C such that $f(x) = C$ for all $x \in (a, b)$.*

Proof. If there is some $d \in (a, b]$ with $f(d) \neq f(a)$, then the Mean Value Theorem says that there is some $c \in (a, d)$ with $f'(c) = (f(d) - f(a))/(d - a) \neq 0$, a contradiction. So, we must have $f(d) = f(a)$ for all $d \in (a, b]$. That is, f is a constant function. □

Corollary 3.22 (**Two Functions with the Same Derivative Differ by a Constant**)**.** *Let $a < b$. Let $f, g: (a, b) \to \mathbb{R}$ be differentiable with $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there is a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.*

Proof. Apply the previous Corollary to the function $h(x) = f(x) - g(x)$. Note that $h'(x) = 0$, so $h(x) = C$ for some constant C. □

Example 3.23. Let's find the function f such that $f'(x) = \cos x$ and such that $f(0) = 2$.

Recall that $(d/dx) \sin x = \cos x$. So, from the Corollary above, we must have $f(x) = \sin x + C$ for some constant C. Since $f(0) = \sin(0) + C = C = 2$, we have $C = 2$. So, $f(x) = \sin(x) + 2$.

Example 3.24. Recall our example of an idealized trajectory. Suppose you throw a baseball vertically in the air, with initial upward velocity v_0 and initial position r_0 (and we neglect air friction). Let $r(t)$ denote the position of the baseball (in meters) at time t (in seconds). We know that the acceleration due to gravity is constant, so that $r''(t) = -9.8$. That is, $(d/dt)r'(t) = -9.8$. The function $-9.8t$ also satisfies $(d/dt)(-9.8t) = -9.8$. Therefore, $r'(t) = -9.8t + C$ for some constant C. Since $r'(0) = v_0 = C$, we conclude that $r'(t) = -9.8t + v_0$. Also, the function $(-9.8/2)t^2 + v_0 t$ satisfies $(d/dt)[(-9.8/2)t^2 + v_0 t] = -9.8t + v_0$. So, there is some constant C_1 such that $r(t) = (-9.8/2)t^2 + v_0 t + C_1$. Since $r(0) = C_1 = r_0$, we have found that

$$r(t) = (-9.8/2)t^2 + v_0 t + r_0.$$

By examining the first and second derivatives of a function, we can say a lot about the properties of that function. In the previous section, we saw that the zeros of the first derivative tell us most of the information about the maximum and minimum values of a given function. To see what else the derivatives tell us, see the JAVA Applet, graph features. Below, we will describe several tests on the first and second derivatives that allow us to find several properties of a function, as in this Applet.

Definition 3.25. Let $f\colon (a,b)\to\mathbb{R}$. If $f(x)<f(y)$ whenever $x<y$ with $x,y\in(a,b)$, we say that f is **increasing** on (a,b). If $f(x)>f(y)$ whenever $x<y$ with $x,y\in(a,b)$, we say that f is **decreasing** on (a,b). If f is increasing on (a,b), or if f is decreasing on (a,b), we say that f is **monotonic**.

Proposition 3.26 (**Increasing/Decreasing Test**). *Let $f\colon (a,b)\to\mathbb{R}$ be differentiable.*

- *If $f'(x)>0$ for all $x\in(a,b)$, then f is increasing on (a,b).*
- *If $f'(x)<0$ for all $x\in(a,b)$, then f is decreasing on (a,b).*

Proof. We prove the first assertion by contradiction. Suppose f is not increasing but $f'>0$ on (a,b). Then there are $x,y\in(a,b)$ with $x<y$ but $f(x)\geq f(y)$. That is, $f(y)-f(x)\leq 0$. By the Mean Value Theorem, there is some $c\in(x,y)$ with $f'(c)=(f(y)-f(x))/(y-x)\leq 0$, a contradiction. We conclude that f is increasing. The second assertion is proven similarly. □

Example 3.27. Consider $f(x)=x^3-3x+1$. We have $f'(x)=3x^2-3=3(x^2-1)=3(x+1)(x-1)$. So, when $x<-1$, $f'(x)>0$ and f is increasing; when $-1<x<1$, $f'(x)<0$ and f is decreasing; and when $x>1$, $f'(x)>0$ and f is increasing.

Proposition 3.28 (**First Derivative Test**). *Let $c\in(a,b)$ be a critical point for a continuous function $f\colon (a,b)\to\mathbb{R}$. Assume that f is differentiable on (a,c) and on (c,b).*

- *If $f'(x)>0$ on (a,c), and if $f'(x)<0$ on (c,b), then f has a local maximum at $x=c$.*
- *If $f'(x)<0$ on (a,c), and if $f'(x)>0$ on (c,b), then f has a local minimum at $x=c$.*
- *If $f'(x)>0$ on $(a,c)\cup(c,b)$, or if $f'(x)<0$ on $(a,c)\cup(c,b)$, then f does not have a local maximum or a local minimum at $x=c$.*

Proof. Suppose $f'(x)>0$ on (a,c) and $f'(x)<0$ on (c,b). Then f is increasing on (a,c) and decreasing on (c,b). So, $f'(c)$ is a local maximum. The other assertions are proven similarly. □

Example 3.29. Consider again $f(x)=x^3-3x+1$. Recall that when $x<-1$, $f'(x)>0$; when $-1<x<1$, $f'(x)<0$; and when $x>1$, $f'(x)>0$. So, $x=-1$ is a local maximum, and $x=1$ is a local minimum.

Example 3.30. Let $f(x)=x^3$. Then $f'(x)=3x^2$. So, $f'(0)=0$, but $f'(x)>0$ for $x\neq 0$. So, 0 is not a local extremum of f.

3.4. **Graph Sketching.**

Definition 3.31 (**Concavity**). Let $f\colon (a,b)\to\mathbb{R}$ be a differentiable function.

- If f' is increasing on (a,b), then f is **concave up**. That is, if $y(x)=ax+b$ denotes a tangent line to f, then $f(x)\geq y(x)$ for all $x\in(a,b)$.
- If f' is decreasing on (a,b), then f is **concave down**. That is, if $y(x)=ax+b$ denotes a tangent line to f, then $f(x)\leq y(x)$ for all $x\in(a,b)$.

Example 3.32. Let $f(x) = x^2$. Then $f'(x) = 2x$ is increasing, so f is concave up.

Let $f(x) = -x^2$. Then $f'(x) = -2x$ is decreasing, so f is concave down.

If f'' exists, and if $f'' > 0$ then f' is increasing, and if $f'' < 0$ then f' is decreasing. We therefore get the following test.

Proposition 3.33 (**Concavity Test**). *Let $f: (a, b) \to \mathbb{R}$ be a differentiable function such that f'' exists on (a, b).*

- *If $f''(x) > 0$ on (a, b), then f is concave up on (a, b)*
- *If $f''(x) < 0$ on (a, b), then f is concave down on (a, b)*

Definition 3.34 (**Inflection Point**). Let $f: (a, b) \to \mathbb{R}$ be a differentiable function and let $c \in (a, b)$. If f is concave up on one side of c, and if f is concave down on the other side of c, then c is called an **inflection point**. If f'' exists on (a, b), then an inflection point occurs when f'' changes sign.

Example 3.35. Let $f(x) = x^3$. Then $f''(x) = 6x$, so f'' changes sign at $x = 0$. That is, $x = 0$ is an inflection point of f. Also, $f''(x) < 0$ when $x < 0$, so f is concave down when $x < 0$. Similarly, $f''(x) > 0$ when $x > 0$ so f is concave up when $x > 0$.

Proposition 3.36 (**Second Derivative Test**). *Let $f: (a, b) \to \mathbb{R}$. Let $c \in (a, b)$. Assume that $f'(c)$ and $f''(c)$ exist. Assume also that $f'(c) = 0$, and that f'' is continuous near c.*

1. *If $f''(c) > 0$, then f has a local minimum at c.*
2. *If $f''(c) < 0$, then f has a local maximum at c.*
3. *If $f''(c) = 0$, then f may or may not have a local extremum at c.*

Example 3.37. Let $f(x) = x^2$. Then $f'(0) = 0$ and $f''(0) = 2 > 0$. So, f has a local minimum at $x = 0$.

Let $f(x) = -x^2$. Then $f'(0) = 0$ and $f''(0) = -2 < 0$. So, f has a local maximum at $x = 0$.

The functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$ all satisfy $f'(0) = 0$ and $f''(0) = 0$, though they have a local minimum, maximum and neither, respectively.

Proof. We only prove (1), since (2) is proven similarly. Since f'' is continuous near c, the definition of continuity says that there is a small interval containing c where $f''(x) > 0$. Then the Concavity Test, Proposition 3.33, says that $f(c + h) \geq f'(c)h + f(c)$ for all sufficiently small h. Since $f'(c) = 0$, we have $f(c + h) \geq f(c)$, so c is a local minimum of f. □

Exercise 3.38. Consider the function $f(x) = x^3/3 - x - 1$. Identify all local maxima, minima and inflection points. Identify where f is increasing and decreasing. Identify where f is concave up and concave down. Then, sketch the function f.

We have $f'(x) = x^2 - 1 = (x + 1)(x - 1)$ and $f''(x) = 2x$. So, $x = 0$ is an inflection point and $x = 1, -1$ are critical points. The function f is increasing when $x < -1$, decreasing when $-1 < x < 1$ and increasing when $x > 1$. So, $x = -1$ is a local maximum and $x = 1$ is a local minimum. Alternatively, $f''(-1) < 0$ and $f''(1) > 0$, which again implies that $x = -1$ is a local max and $x = 1$ is a local min. Lastly, $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$. So, f is concave down when $x < 0$, and f is concave up when $x > 0$.

Definition 3.39 (**Asymptotes**). Let f be a function and let L be a constant. A horizontal line $y = L$ is called a **horizontal asymptote** of f if $\lim_{x \to \infty} f(x) = L$ or if $\lim_{x \to -\infty} f(x) =$

L. A vertical line $x = L$ is called a **vertical asymptote** if $\lim_{x\to L^+} f(x) = \pm\infty$ or if $\lim_{x\to L^-} f(x) = \pm\infty$.

Example 3.40. Let's identify all asymptotes of the function $f(x) = 1/x$.

We have $\lim_{x\to\pm\infty} f(x) = 0$. Therefore, $y = 0$ is a horizontal asymptote of f. Also, $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$. So, $x = 0$ is a vertical asymptote. At all other points, f is a bounded function, so these are the only asymptotes of f.

To sketch the graph of the function f, note that $f'(x) = -x^{-2}$ and $f''(x) = 2x^{-3}$. So, f is concave up when $x > 0$ and concave down when $x < 0$.

3.5. **Applied Optimization.** Optimization problems occur in all applications of mathematics. Here is a simplified example.

Example 3.41. Suppose we want to design a cylindrical soda can with a minimal amount of material. The can's volume is 1 liter (1000 cm^3), and it will be made from aluminum of a fixed thickness. What dimensions should the can have?

Suppose the can has radius r and height h where $r, h > 0$. Then the volume of the can is $\pi r^2 h$. And the surface area of the can is $2\pi r^2 + 2\pi rh$. So, we have $\pi r^2 h = 1000$, or $h = 1000/(\pi r^2)$. And we want to minimize the surface area

$$f(r) = 2\pi r^2 + 2000/r, \qquad r > 0.$$

We look for critical points of f. We have $f'(r) = 4\pi r - 2000r^{-2}$. So, $f'(r) = 0$ when $4\pi r = 2000r^{-2}$, i.e. when $r^3 = 500/\pi$, so that $r = (500/\pi)^{1/3}$. So, a critical point occurs with radius $r = (500/\pi)^{1/3}$ and height $h = 1000/(\pi r^2) = 1000/(\pi^{1/3}(500)^{2/3}) = 2(500/\pi)^{1/3}$. That is, the critical soda can has a height which is twice its radius.

Note that $f'(r) < 0$ when $0 < r < (500/\pi)^{1/3}$ and $f'(r) > 0$ when $r > (500/\pi)^{1/3}$. So, the value $r = (500/\pi)^{1/3}$ is an absolute minimum of f.

Note also that this optimization problem has no absolute maximum, since $\lim_{r\to 0} f(r) = \infty$ and $\lim_{r\to\infty} f(r) = \infty$.

An algorithm for solving optimization problems can be described as follows.

Algorithm 1.

- Introduce variables, and introduce a function f to be optimized.
- Identify the domain of the optimization. Then, apply our usual optimization procedure:
- Find critical points of f in the domain of f.
- Test the critical points of f, and check the endpoints of the domain of f.
- Choose the largest and smallest values of f from these points.

Example 3.42. Find two numbers which sum to 50 and whose product is a maximum.

Given two numbers x, y such that $x + y = 50$, we want to maximize the product xy. Since $y = 50 - x$, we want to maximize $f(x) = xy = x(50 - x)$ over all of $x \in \mathbb{R}$. We check for critical points. We have $f(x) = -x^2 + 50x$, so $f'(x) = -2x + 50$. And $f'(x) = 0$ when $x = 25$. So, the only critical points occurs when $x = 25$. At this point, we have $y = 50 - x = 25$. Note that $f'(x) > 0$ when $x < 25$ and $f'(x) < 0$ when $x > 25$. So, f has an absolute maximum at $x = 25$, and it is unnecessary to check the endpoints of the domain. Note however that $\lim_{x\to\pm\infty} f(x) = -\infty$, so f has no absolute minimum.

3.6. **Newton's Method.** From Algorithm 1, we see that most of the work of optimizing a function involves finding x such that $f'(x) = 0$. In practice, the function $f'(x)$ is sometimes too complicated, and the equation $f'(x) = 0$ may be difficult to solve for x. Thankfully, Newton came up with a general method for finding the zeros of general differentiable functions. This procedure does not work all the time, but it works enough of the time that it is quite useful. It is used by your calculator, for example, whenever you try to find the zeros of a function.

Algorithm 2. Newon's Method, a general way to find the roots of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$.

(1) Choose any point $x_0 \in \mathbb{R}$.
(2) Compute the tangent line of f at x_0: $y(x) = f'(x_0)(x - x_0) + f(x_0)$.
(3) Find x_1 such that $y(x_1) = 0$. This is the intersection of the tangent line $y(x)$ with the x-axis. Note that x_1 satisfies

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

(4) Return to step (2), but replace x_0 with x_1. At the n^{th} iteration of the algorithm, compute the tangent line of f at x_n in step (2), and then find an x_{n+1} in step (3) which is a zero of the tangent line. So, in general we iterate the following equation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 3.43. Let's find the first few iterations of Newton's Method when $f(x) = x^2 - 3$ with $x_0 = 1$. Then $f'(x) = 2x$, and

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{2} = 2.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1}{4} = 7/4.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 7/4 - \frac{f(7/4)}{f'(7/4)} = 7/4 - \frac{1/16}{7/2} = 7/4 - 1/56 = 97/56.$$

And so on. It looks like x_n is getting close to the positive zero of f. That is, x_n is getting close to a value $x > 0$ where $f(x) = 0$, i.e. where $x^2 = 3$, i.e. where $x = \sqrt{3} \approx 1.7320508...$. Indeed, already at the third iteration we have $x_3 = 97/56 \approx 1.73214...$

Remark 3.44. To see an illustration of Newton's Method, see the Applet, Newton Example. In many examples, it only takes a few iterations of the algorithm to get a good approximation for a zero of f.

Remark 3.45. If we ever find a point in step (2) where $f' = 0$, the algorithm will be unable to continue. Actually, if we repeatedly encounter points where f' is close to zero, then Newton's Method will not work very well. There are a few ways to adjust the function f or the starting value x_0 to deal with these issues, so that a suitable modification of Algorithm 2 can find the zeros of many functions.

4. The Integral

Along with the derivative, the integral is one of the two most fundamental concepts that we find in Calculus. Unfortunately, the formal definition of the integral is more complicated than that of the derivative. However, we should still try to understand these formal definitions, since the ideas that go into the construction of the derivative and the integral are pervasive throughout mathematics and the sciences. In the case of the integral, the quantity $\int_a^b f(x)dx$ intuitively represents the area under the curve $y = f(x)$ on the interval $[a, b]$ (if f is positive on the interval $[a, b]$).

Ultimately, we want to find the area under any given curve. The strategy is similar in spirit to our construction of the derivative. We would like to perform some process that requires infinitely many steps, and as noted by Zeno, doing so does not make any sense. To resolve this issue, we approximate some infinite thing by a finite number of steps. And we hope that, as our approximation gets "finer," some number will approach some limit.

Using this paradigm, we first approximate the area under a given curve by a finite number of rectangles. We know the area of a rectangle, so we therefore know the area of several non-overlapping rectangles. We then want to make our approximation of rectangles finer and finer, and then take some limit. If we complete this process in the right way, and if our curve is nice enough, then this limit will exist. Unfortunately, the limit of the sum of the areas of these rectangles may not always exist, so we have to be careful in our construction of the integral. So, although the notation below and the details may appear pedantic or unnecessary, these things really are necessary in order to get a sensible answer in the end.

4.1. **Summation Notation.**
Below, we will be using summation notation often.

Definition 4.1 (Summation Notation). Let $y_1, y_2, \ldots, y_n$ be a set of numbers. We define

$$\sum_{i=1}^{n} y_i = y_1 + y_2 + \cdots + y_{n-1} + y_n.$$

Example 4.2. $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n = n(n+1)/2.$

Example 4.3. $\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6.$

Example 4.4. If $y_i = (-1)^i$, then

$$\sum_{i=1}^{n} y_i = -1 + 1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + (-1)^n = \begin{cases} -1 & \text{, if } n \text{ is odd} \\ 0 & \text{, if } n \text{ is even.} \end{cases}$$

Proposition 4.5 (Properties of Finite Sums). *Let $y_1, \ldots, y_n$ be a set of numbers, and let $z_1, \ldots, z_n$ be a set of numbers. Let c be a constant. Then*

- $\sum_{i=1}^{n}(y_i + z_i) = \left(\sum_{i=1}^{n} y_i\right) + \left(\sum_{i=1}^{n} z_i\right).$
- $\sum_{i=1}^{n}(y_i - z_i) = \left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} z_i\right).$
- $\sum_{i=1}^{n} c y_i = c \sum_{i=1}^{n} y_i.$
- $\sum_{i=1}^{n} c = cn.$

Example 4.6. $\sum_{i=1}^{n}(3i - i^2) = 3\left(\sum_{i=1}^{n} i\right) - \left(\sum_{i=1}^{n} i^2\right).$

4.2. **The Definite Integral.** Let $a < b$. Let $f\colon [a,b] \to \mathbb{R}$ be a positive function. We would like to compute the area under the curve of f on the interval $[a,b]$. We will eventually do this, but for now we will settle for an approximation. We will first approximate this area by a set of rectangles.

Definition 4.7 (**Riemann Sums**)**.** Let $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. The **Riemann sum** of f on $[a,b]$ evaluated at the right endpoints of the rectangles is the quantity

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(x_i).$$

The **Riemann sum** of f on $[a,b]$ evaluated at the left endpoints is the quantity

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(x_{i-1}).$$

The **Riemann sum** of f on $[a,b]$ evaluated at the midpoints is the quantity

$$\sum_{i=1}^{n}(x_i - x_{i-1})f\left(\frac{x_{i-1}+x_i}{2}\right).$$

In each case, we are approximating the area under the curve f by a set of rectangles. For the Riemann sum evaluated at the right endpoints, the quantity $(x_i - x_{i-1})$ is the width of the i^{th} rectangle, and the quantity $f(x_i)$ is the height of the i^{th} rectangle, where $1 \leq i \leq n$.

Each of these Riemann sums provides a good approximation to the area under the curve of f when n is large. In fact, some of these Riemann sums even have a limiting value as $n \to \infty$, if we make the right choices for the points $x_0, \ldots, x_n$.

Example 4.8. Let $f(x) = x$ and consider the interval $[0,1]$. We know that the area under the curve of f on $[0,1]$ is a triangle of area $1/2$. Let's show that the Riemann sums defined above converge to $1/2$ as $n \to \infty$, if we choose $x_0 = 0$, $x_1 = 1/n$, $x_2 = 2/n$, $x_3 = 3/n$, …, $x_{n-1} = (n-1)/n$, $x_n = 1$. That is, we have $x_i = i/n$ for all $i \in \{1, \ldots, n\}$. With these choices, we always have $x_i - x_{i-1} = i/n - (i-1)/n = 1/n$, for any $i \in \{1, \ldots, n\}$. So, the Riemann sum evaluated at the right endpoints is equal to

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(x_i) = \sum_{i=1}^{n}\frac{1}{n}f(i/n) = \frac{1}{n}\sum_{i=1}^{n}f(i/n) = \frac{1}{n}\sum_{i=1}^{n}i/n = \frac{1}{n}\frac{n(n+1)}{2n} = \frac{n+1}{2n}.$$

So,

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - x_{i-1})f(x_i) = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}.$$

Similarly,

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - x_{i-1})f(x_{i-1}) = \frac{1}{2}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - x_{i-1})f\left(\frac{x_{i-1}+x_i}{2}\right) = \frac{1}{2}.$$

That is, each Riemann sum approaches the area under the curve, as $n \to \infty$. More precisely, the Riemann sum approaches the area under the curve, as the maximum spacing between the points goes to zero. That is, as $\max_{i=1,\ldots,n}(x_i - x_{i-1}) \to 0$, the Riemann sum approaches the area under the curve.

Example 4.9. Let $f(x) = x^2$ and consider the interval $[0, 1]$. Let's show that the Riemann sums defined above converge as $n \to \infty$, if we choose $x_0 = 0$, $x_1 = 1/n$, $x_2 = 2/n$, $x_3 = 3/n$, $\ldots$, $x_{n-1} = (n-1)/n$, $x_n = 1$. That is, we have $x_i = i/n$ for all $i \in \{1, \ldots, n\}$. With these choices, we always have $x_i - x_{i-1} = i/n - (i-1)/n = 1/n$, for any $i \in \{1, \ldots, n\}$. So, the Riemann sum evaluated at the right endpoints is equal to

$$\begin{aligned}\sum_{i=1}^{n}(x_i - x_{i-1})f(x_i) &= \sum_{i=1}^{n}\frac{1}{n}f(i/n) = \frac{1}{n}\sum_{i=1}^{n}f(i/n) = \frac{1}{n}\sum_{i=1}^{n}i^2/n^2\\ &= \frac{1}{n}\frac{n(n+1)(2n+1)}{6n^2} = \frac{(n+1)(2n+1)}{6n^2}.\end{aligned}$$

So,

$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - x_{i-1})f(x_i) = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} = \frac{2}{6} = \frac{1}{3}.$$

That is, as $\max_{i=1,\ldots,n}(x_i - x_{i-1}) \to 0$, the Riemann sum approaches the the value $1/3$, which is presumably the area under the curve of f on the interval $[0, 1]$.

We recommend seeing this picture in action with the help of the JAVA applet, Riemann sums.

Definition 4.10 (Riemann Sums). A general **Riemann sum** of f on $[a, b]$ is defined as follows. Let $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. Let $c_1 \in [x_0, x_1]$, $c_2 \in [x_1, x_2]$, $\ldots$, $c_n \in [x_{n-1}, x_n]$. A general Riemann sum is any sum of the form

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(c_i).$$

Example 4.11. Choosing $c_i = x_i$, or $c_i = x_{i-1}$ or $c_i = (x_{i-1}+x_i)/2$ yields the right endpoint, left endpoint, and midpoint Riemann sums, respectively.

Definition 4.12. Let $a < b$, and let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The maximum width of the rectangles of the Riemann sum is denoted by

$$\max_{i=1,\ldots,n}(x_i - x_{i-1}).$$

This number is the maximum of the numbers $(x_1 - x_0), (x_2 - x_1), (x_3 - x_2), \ldots, (x_n - x_{n-1})$. If $\max_{i=1,\ldots,n}(x_i - x_{i-1})$ is small, then the partition is very fine. More specifically, all of our approximating rectangles will have small width. In order to construct the integral, we will let $\max_{i=1,\ldots,n}(x_i - x_{i-1})$ approach zero.

We can finally define the Definite Integral.

Definition 4.13 (**The Definite Integral**)**.** Let $f\colon [a,b]\to\mathbb{R}$. If the following limit exists, we say that f is integrable on $[a,b]$.

$$\int_a^b f(x)dx = \lim_{\left(\max\limits_{i=1,\ldots,n}(x_i-x_{i-1})\right)\to 0}\sum_{i=1}^n (x_i-x_{i-1})f(c_i).$$

Remark 4.14. We now describe the limit appearing in the Definite Integral more explicitly. For $\lim_{\left(\max\limits_{i=1,\ldots,n}(x_i-x_{i-1})\right)\to 0}\sum_{i=1}^n (x_i-x_{i-1})f(c_i)$ to exist and be equal to I, we mean the following. For every $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that: for any choices of $a=x_0<x_1<\cdots<x_n=b$, and for any choices of $c_i\in[x_{i-1},x_i]$, as long as $\max_{i=1,\ldots,n}(x_i-x_{i-1})<\delta$, we have

$$\left|\sum_{i=1}^n (x_i-x_{i-1})f(c_i)-I\right|<\varepsilon.$$

That is, for the limit $\lim_{\left(\max\limits_{i=1,\ldots,n}(x_i-x_{i-1})\right)\to 0}\sum_{i=1}^n (x_i-x_{i-1})f(c_i)$ to exist, we require that any sufficiently fine partition has a Riemann sum that is close to the value I.

Remark 4.15.

- We refer to $\int_a^b f(x)dx$ as the **integral** of f on $[a,b]$.
- The function f inside the integral is called the **integrand**.
- The numbers a,b representing the interval $[a,b]$ are called the **limits of integration**. In the integral $\int_a^b f(x)dx$, the variable x is just a placeholder, which has no intrinsic meaning. For example, we could just as easily write the integral of f on $[a,b]$ as $\int_a^b f(z)dz$ or $\int_a^b f(s)ds$.

Remark 4.16 (**Geometric Interpretation of the Integral**)**.** If $f\colon[a,b]\to\mathbb{R}$ has $f(x)\geq 0$ for all $x\in[a,b]$, then $\int_a^b f(x)dx$ represents the area under the curve of f. If $f\colon[a,b]\to\mathbb{R}$ has some negative values, then $\int_a^b f(x)dx$ represents the **signed area** under the curve of f. That is, $\int_a^b f(x)dx$ is the area enclosed by f lying above the x-axis, minus the area enclosed by f lying below the x-axis.

Example 4.17. Let $f(x)=x$. Then $\int_{-1}^1 f(x)dx=0$, since the area of f above the x-axis is a triangle of area $1/2$, and the area of f below the x-axis is also a triangle of area $1/2$, so $\int_{-1}^1 f(x)dx = 1/2-1/2=0$.

For a function $f\colon[a,b]\to\mathbb{R}$, we do not yet have a way to determine whether or not $\int_a^b f(x)dx$ exists. Thankfully, if $f\colon[a,b]\to\mathbb{R}$ is continuous, then $\int_a^b f(x)dx$ exists, as the following very important theorem shows. However, there are situations where the integral of a function does not exist. We will investigate these situations more below. If we understand when integrals do not exist, then our understanding of the integral is improved, just as an understanding of nonexistence of derivatives improves our understanding of derivatives.

Theorem 4.18 (**Continuous Functions on Closed Intervals are Integrable**)**.** *Let $a<b$. Let $f\colon[a,b]\to\mathbb{R}$ be continuous. Then $\int_a^b f(x)dx$ exists.*

Remark 4.19. We should also mention that the closed interval condition is crucial in Theorem 4.18. For example, $f(x) = 1/x$ is continuous on $(0, 1)$, but $\int_0^1 f(x)dx$ does not exist.

Since a composition of continuous functions is continuous, and a product of continuous functions is continuous, we have the following Corollary of Theorem 4.18.

Corollary 4.20. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be continuous. Then $\int_a^b f(g(x))dx$ exists, and also $\int_a^b f(x)g(x)dx$ exists.*

Proposition 4.21 (Properties of the Definite Integral). *Let $a, b, c, k \in \mathbb{R}$, $a < b < c$. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be integrable on any closed interval.*

(1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
(2) $\int_a^a f(x)dx = 0$.
(3) $\int_a^b k\,dx = k(b-a)$.
(4) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
(5) $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.
(6) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
(7) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.
(8) *If* $f \geq 0$, *then* $\int_a^b f(x)dx \geq 0$.
(9) *If* $f \geq g$, *then* $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
(10) *If* $m \leq f \leq M$, *then* $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.
(11) $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|\,dx$.
(12) $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n} f\left(\frac{i}{n}\right)$
(13) $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{b-a}{n}\right) f\left(a + \frac{i(b-a)}{n}\right)$.

Remark 4.22. Property (12) can be used to evaluate certain infinite sums.

All of these properties can be proven in similar ways, by going back to the definition of the integral. Let's just prove property (4) for the sake of illustration.

Proof sketch of (4). Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and let $c_i \in [x_{i-1}, x_i]$ for each $i \in \{1, \ldots, n\}$. Then

$$\sum_{i=1}^n (x_i - x_{i-1})(f(c_i) + g(c_i)) = \sum_{i=1}^n (x_i - x_{i-1})f(c_i) + \sum_{i=1}^n (x_i - x_{i-1})g(c_i).$$

So, letting $\max_{i=1,\ldots,n}(x_i - x_{i-1}) \to 0$, and using the limit law for sums,

$$\int_a^b (f(x) + g(x))dx = \lim_{\left(\max_{i=1,\ldots,n}(x_i - x_{i-1})\right) \to 0} \sum_{i=1}^n (x_i - x_{i-1})(f(c_i) + g(c_i))$$

$$= \lim_{\left(\max_{i=1,\ldots,n}(x_i - x_{i-1})\right) \to 0} \sum_{i=1}^n (x_i - x_{i-1})f(c_i) + \lim_{\left(\max_{i=1,\ldots,n}(x_i - x_{i-1})\right) \to 0} \sum_{i=1}^n (x_i - x_{i-1})g(c_i)$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx.$$

□

Example 4.23. Suppose $\int_0^2 f(x)dx = 3$ and $\int_0^2 g(x)dx = -2$. Then $\int_0^2 (f(x) + g(x))dx = 3 - 2 = 1$, $\int_0^2 (2f(x) - g(x))dx = 2 \cdot 3 - (-2) = 8$ and $\int_2^0 f(x)dx = -3$.

Example 4.24. Let's use property (10) to estimate the integral $\int_0^2 \sqrt{1 + \sin(x)}dx$. Since $-1 \leq \sin(x) \leq 1$, we have $0 \leq 1 + \sin x \leq 2$. So, $0 \leq \sqrt{1 + \sin x} \leq \sqrt{2}$, and by property (10), we have

$$0 \leq \int_0^2 \sqrt{1 + \sin x}\, dx \leq \int_0^2 \sqrt{2}dx = 2\sqrt{2}.$$

4.3. **The Indefinite Integral.** So far, we know that a continuous function can be integrated on a closed interval. But we cannot yet compute very many integrals. We now head towards our goal of computing many integrals.

Recall Corollary 3.22: if $f'(x) = g'(x)$ for all $x \in \mathbb{R}$, then there is a constant $C \in \mathbb{R}$ such that $f(x) = g(x) + C$.

Definition 4.25 (**Antiderivative, Indefinite Integral**)**.** Let $f : \mathbb{R} \to \mathbb{R}$. We say that F is an **antiderivative** of f if F is differentiable, and for all $x \in \mathbb{R}$ we have $F'(x) = f(x)$. We then use the notation

$$\int f(x)dx = F(x) + C$$

where C is any constant. We refer to $\int f(x)dx$ as the **indefinite integral** of f.

Remark 4.26. The indefinite integral is not associated to any interval. Also, while the definite integral is a number, the indefinite integral is a function. The definite and indefinite integral are related to each other, as we will see below, but they are not quite the same.

Remark 4.27. Let F, G be antiderivatives of f, so that $F'(x) = G'(x) = f$. From Corollary 3.22, there must be a constant C such that $F(x) = G(x)+C$. So, if we have one antiderivative F of f, then the set of all antiderivatives of f is given by the set of all $F(x) + C$, $C \in \mathbb{R}$.

Example 4.28. Let $f(x) = x^2$. Then the set of all antiderivatives of f is given by $F(x) = (1/3)x^3 + C$, $C \in \mathbb{R}$.

Example 4.29 (**Idealized Trajectories**)**.** Recall our example of idealized trajectories. Suppose I throw a ball straight up in the air at a velocity v_0 m/s, with initial vertical position s_0 meters, ignoring air friction. Suppose the ball has mass m kg. The only acceleration that acts on the ball is a constant acceleration due to gravity, of roughly $a(t) = -9.8$ m/s^2, where t is the time after the ball is thrown, measured in seconds. Taking the antiderivative and using Remark 4.27, the ball must have velocity $v(t) = -9.8t + C$. Since $v(0) = v_0$, we conclude that $v(t) = -9.8t + v_0$. Taking the antiderivative again and using Remark 4.27, the ball must have position $s(t) = -(9.8/2)t^2 + v_0 t + C$. My initial vertical position is s_0, so we conclude that the ball has position

$$s(t) = -4.9t^2 + v_0 t + s_0.$$

For now, antiderivatives may seem a bit strange. Also, if we have a function f, how can we know whether or not an antiderivative exists? It turns out that, if f is continuous, then you can create an antiderivative of f by measuring the areas under the curve f. So, calculating areas under a curve has the "opposite" effect of taking a derivative of a curve. This statement will be made more precise when we state the Fundamental Theorem of Calculus. For now, we give a precise description of how to find the area under a curve, via the Riemann integral.

Theorem 4.30 (**Indefinite Integral of Powers of** x)**.** *Let* $n \neq -1$. *Then*

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Proof. Recall that $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{n+1}{n+1}x^n = x^n$. □

Example 4.31. $\int x^3 dx = x^4/4 + C$. $\quad \int x^{3/2} dx = (2/5)x^{5/2} + C$

Proposition 4.32 (**Properties of the Indefinite Integral**)**.** *Let* c *be a constant and let* $f, g \colon \mathbb{R} \to \mathbb{R}$.

- $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.
- $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$.
- $\int cf(x)dx = c \int f(x)dx$.

Example 4.33.

- $\int \sin x \, dx = -\cos x + C$.
- $\int \cos x \, dx = \sin x + C$.
- $\int \sec^2 x \, dx = \tan x + C$.
- $\int \sec x \tan x \, dx = \sec x + C$.
- $\int e^x dx = e^x + C$.
- $\int e^{cx+d} dx = \frac{1}{c} e^{cx+d} + C$, $c \neq 0$.

4.4. **The Fundamental Theorem of Calculus.** We can finally describe the precise manner in which differentiation and integration "cancel each other out." The following Theorem will also allow us to compute many integrals.

Theorem 4.34 (**Fundamental Theorem of Calculus**)**.** *Let* $a < b$.

(i) *Let* $f \colon [a, b] \to \mathbb{R}$ *be differentiable. Assume also that* $f' \colon [a, b] \to \mathbb{R}$ *is continuous. Then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

(ii) *Let* $f \colon [a, b] \to \mathbb{R}$ *be continuous. For* $x \in (a, b)$ *define* $g(x) = \int_a^x f(t)dt$. *Then* g *is an antiderivative of* f, *i.e.*

$$g'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Remark 4.35. Part (i) of Theorem 4.34 can be used to evaluate many different integrals. For example, we have the following two corollaries.

Corollary 4.36 (**Integrating Powers of** x)**.** *Let* $n \in \mathbb{R}$, $n \neq -1$, $0 < a < b$. *Then*

$$\int_a^b x^n dx = \left[\frac{1}{n+1} x^{n+1}\right]_{x=a}^{x=b} = \frac{1}{n+1}(b^{n+1} - a^{n+1}).$$

Proof. Let $f(x) = (1/(n+1))x^{n+1}$. Note that $f'(x) = x^n$, and then apply the Fundamental Theorem, Theorem 4.34(i) to get

$$\int_a^b x^n dx = \int_a^b f'(x) = f(b) - f(a) = \frac{1}{n+1}(b^{n+1} - a^{n+1}).$$

□

Remark 4.37. What happens if we allow $a = -1$, $b = 1$, $n < 0$?

Example 4.38.

$$\int_0^1 x\,dx = [x^2/2]_{x=0}^{x=1} = 1/2 - 0 = 1/2.$$
$$\int_0^1 x^2\,dx = [x^3/3]_{x=0}^{x=1} = 1/3 - 0 = 1/3.$$
$$\int_1^3 (x^4 - x^{-2})dx = [x^5/5 + x^{-1}]_{x=1}^{x=3} = 3^5/5 + 3^{-1} - 1/5 - 1.$$
$$\int_{-\pi/2}^{\pi/2} \cos(x)dx = [\sin x]_{x=-\pi/2}^{x=\pi/2} = \sin(\pi/2) - \sin(-\pi/2) = 1 - (-1) = 2.$$

Example 4.39.

$$\frac{d}{dx}\int_1^x \cos(t)dt = \cos(x).$$
$$\frac{d}{dx}\int_2^x \frac{1}{1+t^2}dt = \frac{1}{1+x^2}.$$
$$\frac{d}{dx}\int_1^{x^2} \cos(t)dt = 2x\cos(x^2).$$

For the last example, write $\int_1^{x^2}\cos(t)dt = f(g(x))$, where $f(y) = \int_1^y \cos(t)dt$ and $g(x) = x^2$. Then the Chain Rule says $(d/dx)f(g(x)) = f'(g(x))g'(x)$.

Proof of of Theorem 4.34(i). Suppose we have a partition of $[a, b]$. That is, we have

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then

$$\begin{aligned} f(b) - f(a) &= f(x_n) - f(x_0) \\ &= f(x_n) + [-f(x_{n-1}) + f(x_{n-1})] + \cdots + [-f(x_1) + f(x_1)] - f(x_0) \\ &= [f(x_n) - f(x_{n-1})] + [f(x_{n-1}) - f(x_{n-2})] + \cdots + [f(x_2) - f(x_1)] + [f(x_1) - f(x_0)] \\ &= \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] \end{aligned}$$

By the Mean Value Theorem, there exists $c_i \in [x_i, x_{i-1}]$ such that, for $i = 1, \ldots, n$,

$$(x_i - x_{i-1})f'(c_i) = f(x_i) - f(x_{i-1}).$$

Therefore,

$$f(b) - f(a) = \sum_{i=1}^{n}(x_i - x_{i-1})f'(c_i). \qquad (*)$$

The right side of $(*)$ is a Riemann sum for f'. Since f' is continuous, we know that f' is integrable. So, letting $\max_{i=1,\ldots,n}(x_i - x_{i-1}) \to 0$ and applying the definition of the definite integral, $(*)$ becomes our desired equality:

$$f(b) - f(a) = \int_a^b f'(t)dt.$$

□

Proof sketch of Theorem 4.34(ii). We treat the difference quotient directly. Let $h \in \mathbb{R}$. Then

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt.$$

For simplicity, assume that f is differentiable. Then, using the linear approximation of f for values of t near the point x, we have $f(t) \approx f(x) + f'(x)(t-x)$, so

$$\begin{aligned}\frac{g(x+h) - g(x)}{h} &\approx \frac{1}{h}\int_x^{x+h} [f(x) + f'(x)(t-x)]dt = \frac{h}{h}f(x) + f'(x)\frac{1}{h}\int_x^{x+h}(t-x)dt \\ &= f(x) + f'(x)\frac{1}{h}[(t-x)^2/2]_{t=x}^{t=x+h} = f(x) + f'(x)\frac{h^2/2}{h} = f(x) + hf'(x)/2.\end{aligned}$$

So, letting $h \to 0$ we get $\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x)$. □

4.5. **Integration by Substitution.**

Theorem 4.40 (Change of Variables/ Substitution). *Let $a < b$, $c < d$. Let $g\colon [a,b] \to [c,d]$ be differentiable. Also, suppose that $f\colon [c,d] \to \mathbb{R}$ is continuous. Then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

Or, in indefinite form, with $u = g(x)$, we have

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Example 4.41. The following integral may appear difficult if not impossible to evaluate, but Theorem 4.40 allows us to evaluate it. For $x > 0$, let $f(t) = e^t$, and let $g(x) = 1/x$. Applying Theorem 4.40 and then Theorem 4.34(i),

$$\int_1^2 \frac{e^{1/x}}{x^2}dx = -\int_1^2 f(g(x))g'(x)dx = -\int_{g(1)}^{g(2)} e^t dt = -\int_1^{1/2} e^t dt = \int_{1/2}^1 \frac{d}{dt}e^t dt = e - \sqrt{e}.$$

Example 4.42. Let $f(t) = \cos(t)$, and let $g(x) = x^2$. Applying Theorem 4.40 and then Theorem 4.34(i),

$$\begin{aligned}\int_0^{\sqrt{\pi/4}} x\cos(x^2)dx &= \frac{1}{2}\int_0^{\sqrt{\pi/4}} f(g(x))g'(x)dx = \frac{1}{2}\int_{g(0)}^{g(\sqrt{\pi/4})} f(t)dt = \frac{1}{2}\int_0^{\pi/4}\cos(t)dt \\ &= \frac{1}{2}\int_0^{\pi/4}\frac{d}{dt}\sin(t)dt = \frac{1}{2}(\sin(\pi/4) - \sin(0)) = \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}.\end{aligned}$$

Example 4.43. Let $u = x^2 + 1$ so that $du = 2xdx$, i.e. $xdx = (1/2)du$. Then

$$\int \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int u^{-2}du = -\frac{1}{2}u^{-1} = -\frac{1}{2(x^2+1)}.$$

And indeed, we can verify that $-\frac{d}{dx}\frac{1}{2(x^2+1)} = \frac{4x}{(2(x^2+1))^2} = \frac{x}{(x^2+1)^2}$.

Example 4.44. Let $u = x + 1$ so that $du = dx$. Then

$$\int x\sqrt{x+1}dx = \int (u-1)u^{1/2}du = \int u^{3/2} - u^{1/2}du$$
$$= (2/5)u^{5/2} - (2/3)u^{3/2} = (2/5)(x+1)^{5/2} - (2/3)(x+1)^{3/2}.$$

And we can verify that $\frac{d}{dx}[(2/5)(x+1)^{5/2} - (2/3)(x+1)^{3/2}] = (x+1)^{3/2} - (x+1)^{1/2} = (x+1-1)\sqrt{x+1} = x\sqrt{x+1}$.

Proof of Theorem 4.40. For $a < x < b$, define $F(x) = \int_a^x f(t)dt$. From the Fundamental Theorem of Calculus, Theorem 4.34(ii), $F'(x) = f(x)$. Also, by the Chain Rule, $(d/dx)[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$. Note also that $f(g(x))g'(x)$ is integrable by Corollary 4.20. Putting everything together, we have

$$\int_a^b f(g(x))g'(x)dx = \int_a^b \frac{d}{dx}[F(g(x))]dx = F(g(b)) - F(g(a)), \qquad \text{by Theorem 4.34(i)}$$
$$= \int_{g(b)}^{g(a)} \frac{d}{dx}F(x)dx, \qquad \text{by Theorem 4.34(i)}$$
$$= \int_{g(b)}^{g(a)} f(x)dx$$

□

5. Applications of the Integral

5.1. Areas Between Curves.

Definition 5.1. Let $a < b$. Suppose $f, g\colon [a,b] \to \mathbb{R}$ satisfy $f(x) \geq g(x)$ for all $x \in [a,b]$. We define the area between the curves f and g on $[a,b]$ to be

$$\int_a^b (f(x) - g(x))dx.$$

For general functions f, g, we define the area between the curves f and g on $[a,b]$ to be

$$\int_a^b |f(x) - g(x)|\, dx.$$

Example 5.2. Let $f(x) = x$ and let $g(x) = -x$. Then the area between these curves on $[0,1]$ is

$$\int_0^1 x - (-x)dx = \int_0^1 2xdx = [x^2]_{x=0}^{x=1} = 1 - 0 = 1.$$

Example 5.3. Let $f(x) = x$ and let $g(x) = x^2$. Note that $f(x) \geq g(x)$ when $x \in [0, 1]$ and $g(x) \geq f(x)$ when $x \in [-2, 0]$. So, the area between these curves on $[-2, 1]$ is

$$\int_{-2}^{1} \left|x - x^2\right| dx = \int_{0}^{1} (x - x^2)dx + \int_{-2}^{0} (x^2 - x)dx = [x^2/2 - x^3/3]_{x=0}^{x=1} + [x^3/3 - x^2/2]_{x=-2}^{x=0}$$
$$= (1/2) - (1/3) - (-2)^3/3 + (-2)^2/2 = (1/2) - (1/3) + 8/3 + 2 = 29/6.$$

Example 5.4. Let's find the area between the curves $x = 0$ and $x = y^2 + 1$ lying between the lines $y = 0$ and $y = 1$. This area is given by

$$\int_{0}^{1} (y^2 + 1)dy = [(1/3)y^3 + y]_{y=0}^{y=1} = (1/3) + 1 = 4/3.$$

5.2. **Average Value.**

Definition 5.5 (**Average Value**)**.** Let $a < b$. Let $f \colon [a, b] \to \mathbb{R}$. The **average value** of f on the interval $[a, b]$ is defined to be

$$\frac{1}{b - a} \int_{a}^{b} f(x)dx.$$

Example 5.6. The average value of $f(x) = x$ on the interval $[0, 20]$ is

$$\frac{1}{20} \int_{0}^{20} xdx = \frac{1}{20}[x^2/2]_0^{20} = \frac{1}{40}(400) = 10.$$

Example 5.7. The average value of $f(x) = x/(x^2 + 1)^2$ on the interval $[3, 6]$ is (using $u = x^2 + 1$ so $du = 2xdx$, i.e. $xdx = du/2$)

$$\frac{1}{6 - 3} \int_{3}^{6} \frac{x}{(x^2 + 1)^2} dx = \frac{1}{3} \int_{10}^{37} \frac{u^{-2}}{2} du = \frac{1}{6}[-u^{-1}]_{u=10}^{u=37} = \frac{1}{6}\left(-\frac{1}{37} + \frac{1}{10}\right)$$

5.3. **Volumes by Revolution.**

Definition 5.8. A **solid of revolution** is obtained by taking a region in the plane and rotating the region about an axis.

Example 5.9. Consider the region in the plane lying above the x-axis and below the curve $y = \sqrt{1 - x^2}$. If we rotate this region around the x-axis, we obtain the ball of radius 1.

Example 5.10. Consider the region in the plane lying above the x-axis, below the line $y = 1$, and between the lines $x = 0$ and $x = 1$. If we rotate this region around the x-axis, we obtain a circular cylinder of radius 1 and of height 1.

Proposition 5.11 (**Volume of a Solid by Revolution: Disk Method**)**.** *Let $f \colon [a, b] \to \mathbb{R}$ be continuous with $f(x) \geq 0$ for all $x \in [a, b]$. Consider region in the plane lying above the interval $[a, b]$ (on the x-axis) and lying below the curve $y = f(x)$. When this region is rotated around the x-axis, the resulting solid of revolution has volume*

$$\pi \int_{a}^{b} (f(x))^2 dx.$$

In this formula, we think of $f(x)$ as the radius of a thin disk encircling the point x.

Remark 5.12. If c is a constant, and if $f = c$ is a constant function, then the solid of revolution is a cylinder of radius c and height $b - a$. So, the cylinder has volume $\pi c^2(b - a)$. This formula agrees with the integral above

$$\pi \int_a^b (f(x))^2 dx = \pi \int_a^b c^2 dx = \pi c^2(b - a).$$

For a general function f, we can think of f as being roughly constant on small intervals of the form $[x, x + h]$. If f is roughly constant, then the solid produced by revolving f around the interval $[x, x + h]$ is essentially a cylinder of radius $f(x)$ and of height $(x + h) - x = h$. So, the volume of this thin cylinder is $\pi(f(x)^2)h$. Summing up all of the contributions of these small cylinders then gives the integral above (if we intuitively think of h as being equal to dx, an infinitesimally small cylinder height).

Example 5.13. Let $r > 0$. Consider the region in the plane lying above the x-axis and below the curve $f(x) = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$. If we rotate this region around the x-axis, we obtain the ball of radius r, which we know to have volume $(4/3)\pi r^3$. This formula can also be derived as follows:

$$\pi \int_{-r}^r (f(x))^2 dx = \pi \int_{-r}^r (r^2 - x^2) dx = \pi[r^2 x - x^3/3]_{x=-r}^{x=r} = \pi[r^3 - r^3/3 + r^3 - r^3/3] = \pi r^3(4/3).$$

Example 5.14. Let $r_1 > r_2 > 0$. Consider the region in the plane lying above the x-axis, above the curve $g(x) = \sqrt{r_2^2 - x^2}$ and below the curve $f(x) = \sqrt{r_1^2 - x^2}$. If we rotate this region around the x-axis, we obtain the ball of radius r_1, with a ball of radius r_2 removed. So, the volume of this region is

$$(4/3)\pi r_1^3 - (4/3)\pi r_2^3.$$

Example 5.15. We can also revolve regions around the y-axis. Consider the region bounded between the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$. Revolving this region around the y-axis produces a solid cylinder of volume

$$\pi \int_0^2 (1 - 0)^2 dy = \pi \int_0^2 dy = 2\pi.$$

5.4. **Volumes by Cylindrical Shells.**

Proposition 5.16 (Volume of a Solid by Revolution: Cylindrical Shells). *Let $b > a \geq 0$. Let $f\colon [a, b] \to \mathbb{R}$ be continuous with $f(x) \geq 0$ for all $x \in [a, b]$. Consider the region in the plane lying above the interval $[a, b]$ (on the x-axis) and lying below the curve $y = f(x)$. When this region is rotated around the y-axis, the resulting solid of revolution has volume*

$$2\pi \int_a^b x f(x) dx.$$

In this formula, we think of x as the radius of a cylindrical shell, and we think of $f(x)$ as the height of a cylindrical shell.

Remark 5.17. If c is a constant, and if $f = c$ is a constant function, then the solid of revolutions is a cylinder of height c and radius b, with a cylinder of height c and radius a removed. So, the solid has volume $\pi c(b^2 - a^2)$. This formula agrees with the integral above

$$2\pi \int_a^b x f(x) dx = 2\pi \int_a^b xc\, dx = 2\pi c[x^2/2]_{x=a}^{x=b} = \pi c(b^2 - a^2).$$

For a general function f, we can think of f as being roughly constant on a small intervals of the form $[x, x+h]$. If f is roughly constant, then the solid produced by revolving f on $[x, x+h]$ around the y-axis is essentially a cylinder of radius $x+h$ and of height $f(x)$, minus a cylinder of radius x and of height $f(x)$. So, the volume of this cylindrical shell is $\pi f(x)((x+h)^2 - x^2) = \pi f(x)((x^2+2xh+h^2) - x^2) = 2\pi x f(x)h + \pi f(x)h^2$. The second term does not contribute much when h is small, so it can be ignored. Summing up all of the contributions of these cylindrical shells then gives the integral above (if we intuitively think of h as being equal to dx, an infinitesimally small width).

Example 5.18. Let $r > 0$. Consider the region in the plane lying above the x-axis and below the curve $f(x) = \sqrt{r^2 - x^2}$, $0 \leq x \leq r$. If we rotate this region around the x-axis, we obtain one half of a ball of radius r, which we know to have volume $(2/3)\pi r^3$. This formula can also be derived as follows (using $u = r^2 - x^2$, $-du = 2xdx$):

$$2\pi \int_0^r x f(x) dx = 2\pi \int_0^r x\sqrt{r^2 - x^2} dx = -\pi \int_{r^2}^0 \sqrt{u}\, du = \pi[(2/3)u^{3/2}]_{u=0}^{u=r^2} = (2/3)\pi r^3.$$

Example 5.19. Consider the region in the plane lying above interval $[0, r]$ on the x-axis and below the curve $y = h > 0$. If we rotate this region around the y-axis, we obtain a cylinder of radius r and height h. So, the volume of this region is

$$2\pi \int_0^r x f(x) dx = 2\pi \int_0^r x h\, dx = 2\pi h[x^2/2]_{x=0}^{x=r} = \pi r^2 h.$$

Example 5.20. We can also revolve regions around other axes. Consider the region bounded by the curves $x = 0$ and $y = 1 - x^2$. When we revolve this region around the axis $x = 1$ for points $-1 \leq x \leq 1$, the radius of the cylindrical shells will be $(1 - x)$, and their height will be $(1 - x^2)$. So, the solid has volume

$$2\pi \int_{-1}^1 (1-x)(1-x^2) dx = 2\pi \int_{-1}^1 (1 - x^2 - x + x^3) dx = 2\pi[x - x^3/3 - x^2/2 + x^4/4]_{x=-1}^{x=1}$$
$$= 2\pi[1 - (-1) - 1/3 - (-1/3)] = 8\pi/3.$$

6. Appendix: Notation

$\mathbb{R}$ denotes the set of real numbers

$\in$ means "is an element of." For example, $2 \in \mathbb{R}$ is read as "2 is an element of $\mathbb{R}$."

$f\colon A \to B$ means f is a function with domain A and range B. For example,

$f\colon [0,1] \to \mathbb{R}$ means that f is a function with domain $[0,1]$ and range $\mathbb{R}$

Let $f\colon \mathbb{R} \to \mathbb{R}$, and let $a, b, x \in \mathbb{R}$ with $a < b$. Let n be a positive integer.

$\lim_{x\to a} f(x)$ denotes the limit of $f(x)$ as x approaches a

$$f'(x) = \frac{df(x)}{dx} = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ denotes}$$

the derivative of f with respect to x

$f''(x) = \frac{d}{dx}f'(x)$ denotes the second derivative of f with respect to x

$f^{(n)}(x)$ denotes the n^{th} derivative of f with respect to x

$\int_a^b f(x)\,dx$ denotes the definite integral of f on the interval $[a,b]$

$\int f(x)\,dx$ denotes the indefinite integral of f

Remark 6.1. In the book, there are several expressions of the form

$$f(x) = \frac{\cos 2x - x}{x^2}$$

When the cosine is written without parentheses in the argument, it is usually understood that the very first thing that is written after the cosine (in this case $2x$) is the argument of the cosine. The next minus or plus sign that appears is assumed to occur outside of the parentheses (that have been omitted). That is, the expression above can be equivalently written as follows

$$f(x) = \frac{\cos(2x) - x}{x^2}$$

Remark 6.2. The following two expressions are equal:

$$\cos^2(x) = (\cos(x))^2.$$

Remark 6.3. Whenever a fraction has a radical of a number in the denominator, we prefer to move the radical to the numerator, as in the following example:

$$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

If a variable occurs in the radical, then we usually leave the radical in the denominator.

UCLA Department of Mathematics, Los Angeles, CA 90095-1555
E-mail address: heilman@math.ucla.edu