Please provide complete and well-written solutions to the following exercises.

Due March 5, in the discussion section.

Assignment 7

Exercise 1. Prove the following assertions.

(a) (Continuous periodic functions are bounded.) If $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, then f is bounded. (That is, given $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$.)
(b) (Continuous periodic functions form a vector space and an algebra.) Let $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. Then $f+g$, $f-g$ and fg are all in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. Also, if $c \in \mathbf{C}$, then $cf \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.
(c) (Uniform limits of continuous periodic functions are continuous periodic.) Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ which converges uniformly to a function $f\colon \mathbf{R} \to \mathbf{C}$. Then $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

(Hint: for (i), first show that f is bounded on $[0,1]$.)

Exercise 2. Let $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. We define the (complex) **inner product** $\langle f, g\rangle$ to be the quantity

$$\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}dx.$$

Verify that this inner product on $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ satisfies the axioms of a complex inner product space.

Exercise 3. Let $M > 0$ be any positive real number. Find a function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ such that $\|f\|_2 \leq 1$ but such that $\|f\|_\infty > M$. On the other hand, if $g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, show that $\|g\|_2 \leq \|g\|_\infty$. So, the L_2 and sup-norms on $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ are related, but they can also be very different.

Exercise 4. Let $(f_j)_{j=1}^{\infty}$ be a sequence of functions in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, and let f be another function in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

- Show that if $(f_j)_{j=1}^{\infty}$ converges uniformly to f, then $(f_j)_{j=1}^{\infty}$ also converges to f in the L_2 metric.
- Find a sequence $(f_j)_{j=1}^{\infty}$ in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ which converges to some $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ in the L_2 metric, so that $(f_j)_{j=1}^{\infty}$ does not converge to f uniformly. (Hint: consider $f = 0$ and use Exercise 3.)
- Find a sequence $(f_j)_{j=1}^{\infty}$ in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ which converges to some $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ in the L_2 metric, so that $(f_j)_{j=1}^{\infty}$ does not converge pointwise to f. (Hint: consider $f = 0$ and try to make the functions f_j large at one point.)

- Find a sequence $(f_j)_{j=1}^{\infty}$ in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ which converges to some $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ pointwise, so that $(f_j)_{j=1}^{\infty}$ does not converge to f in the L_2 metric. (Hint: consider $f=0$ and try to make the functions f_j large in L_2 norm.)

Exercise 5 (**Characters are an Orthonormal System**)**.** Let n, m be integers. Prove the following. If $n=m$, then $\langle e_n, e_m\rangle = 1$. If $n \neq m$, then $\langle e_n, e_m\rangle = 0$. Also, $\|e_n\|_2 = 1$.

Exercise 6. Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Show that, for all integers $-N \leq n \leq N$, we have

$$c_n = \langle f, e_n\rangle.$$

Also, for any integer n with $|n| > N$, we have $\langle f, e_n\rangle = 0$. And, we have the identity

$$\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2 .$$

(Hint: for the final identity, use either the Pythagorean Theorem and induction, or substitute $f = \sum_{n=-N}^{N} c_n e_n$ into $\|f\|_2^2$ and expand out all of the terms.)

Exercise 7. Let $f, g, h \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. Prove the following assertions.

(a) The convolution $f * g$ is continuous and $\mathbf{Z}$-periodic. That is, $f * g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.
(b) $f * g = g * f$.
(c) $f*(g+h) = f*g + f*h$ and $(f+g)*h = f*h + g*h$. For any complex number c, we have $c(f*g) = (cf)*g = f*(cg)$.

(Hints: to prove (a), you may need to use the uniform continuity of f and the boundedness of g, or vice versa. To prove $f*g = g*f$, you may need to use periodicity to "cut and paste" the interval $[0,1]$.)

Exercise 8. Let $f, g, h \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. For any $n \in \{0,1,2,\ldots\}$, define the **trigonometric Fourier coefficients** a_n, b_n by

$$a_n := 2\int_0^1 f(x)\cos(2\pi n x)dx, \qquad b_n := 2\int_0^1 f(x)\sin(2\pi n x)dx.$$

- Show that the series
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(2\pi n x) + b_n \sin(2\pi n x))$$
converges to f with respect to the L_2 metric. (Hint: use the Fourier inversion theorem, and break up the exponentials into sines and cosines. Combine the positive n terms with the negative n terms.)
- Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then the above series actually converges uniformly to f. (Hint: use the Theorem concerning uniform convergence of Fourier series.)
- For any $x \in [0,1)$, define $f(x) := (1-2x)^2$, and extend f to be $\mathbf{Z}$-periodic on the rest of the real line. Using the second part of this exercise, show that
$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\cos(2\pi n x)$$

converges uniformly to f.

- Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (Hint: let $x = 0$ in the third part of the exercise.)
- Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (Hint: expand the cosines in terms of exponentials, and then use Plancherel's Theorem.)